Senior Kangaroo

Past Papers and Solutions 2011-2014

United Kingdom Mathematics Trust

SENIOR ‘KANGAROO’ MATHEMATICAL CHALLENGE

Friday 2nd December 2011

Organised by the United Kingdom Mathematics Trust

The Senior Kangaroo paper allows students in the UK to test themselves on questions set for the best school-aged mathematicians from across Europe and beyond.

RULES AND GUIDELINES (to be read before starting):

1. Do not open the paper until the Invigilator tells you to do so.
2. Time allowed: **1 hour**.
3. The use of rough paper is allowed; **calculators** and measuring instruments are **forbidden**.
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5. The paper contains 20 questions. Five marks will be awarded for each correct answer. There is no penalty for giving an incorrect answer.
6. The questions on this paper challenge you **to think**, not to guess. Though you will not lose marks for getting answers wrong, you will undoubtedly get more marks, and more satisfaction, by doing a few questions carefully than by guessing lots of answers.

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Maths Challenges Office, School of Maths Satellite,

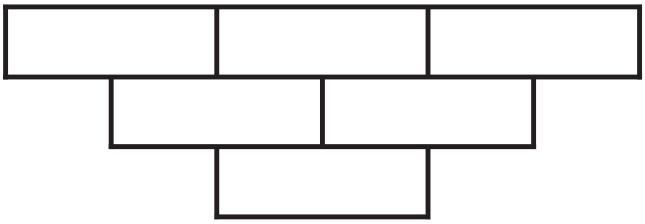
University of Leeds, Leeds, LS2 9JT

Tel. 0113 343 2339

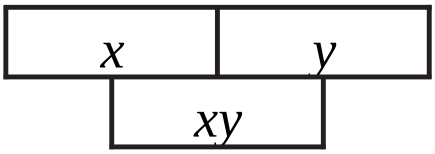
www.ukmt.org.uk

1. The diagram below is to be completed so that:
    a) each cell contains a positive integer;
    b) apart from the top row, the number in each cell is the product of the numbers in the two cells immediately above;
    c) the six numbers are all different.

What is the smallest possible total of the six numbers?



RULE:



2. The mean number of students accepted by a school in the four years 2007 to 2010 was 325. The mean number of students accepted by the school in the five years 2007 to 2011 was 4% higher.

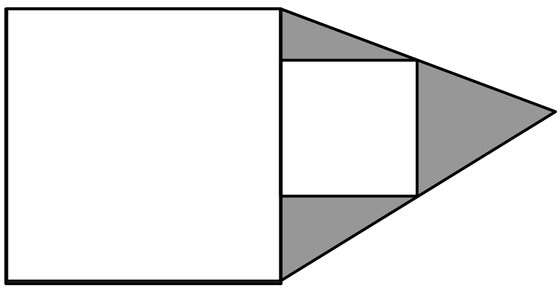
How many students did this school accept in 2011?

3. 200 people stand in a line. The prize-giver walks along the line 200 times, always starting at the same end. On the first pass, the prize-giver gives each person a pound coin. On the second pass along the line, the prize-giver gives every second person another pound. On the third pass, every third person is given another pound, and so on.

After 200 passes, how many pounds has the 120th person been given?

4. The diagram below includes two squares: one has sides of length 20 and the other has sides of length 10.

What is the area of the shaded region?

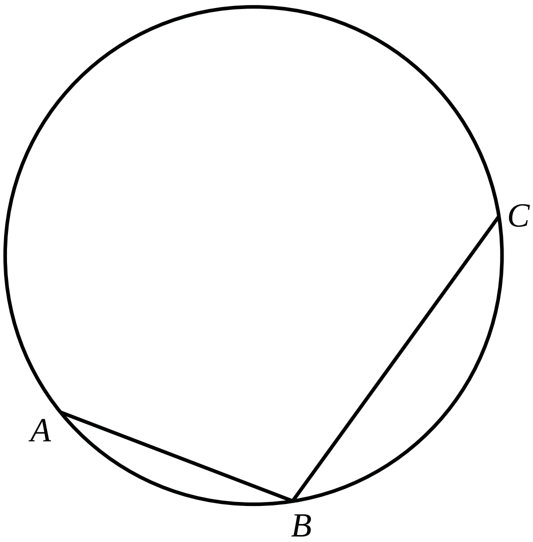


5. How many positive two-digit numbers are there whose square and cube both end in the same digit?

6. The lengths of two sides of an acute-angled triangle and the perpendicular height from the third side of the triangle are 12, 13 and 15 (possibly not in that order).

What is the area of the triangle?

7. In the diagram, the radius of the circle is equal to the length *AB*.
What is the size of angle *ACB*, in degrees?

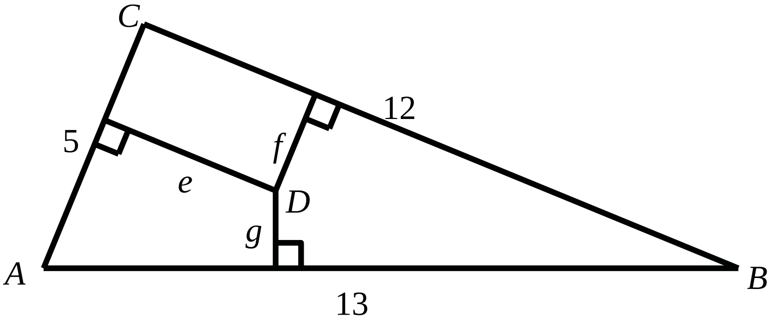


8. The price of an item in pounds and pence is increased by 4%. The new price is exactly *n* pounds where *n* is a whole number.
What is the smallest possible value of *n*?

9. How many squares have $(-1, -1)$ as a vertex and at least one of the coordinate axes as an axis of symmetry?

10. What is the value of $\left(\sqrt{8 + 2\sqrt{7}} - \sqrt{8 - 2\sqrt{7}}\right)^2$?

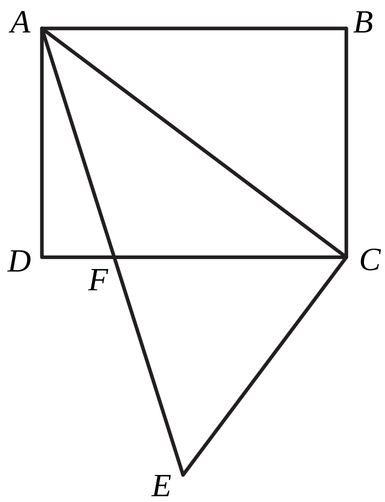
11. In the diagram, *ABC* is a triangle with sides $AB = 13$, $BC = 12$ and $AC = 5$. The point *D* is any point inside the triangle with $CD = 4$ and the perpendicular distances from *D* to the sides of the triangle are *e*, *f* and *g*, as shown.
What is the value of $5e + 12f + 13g$?



12. Elections in Herbville were held recently. Everyone who voted for the Broccoli Party had already eaten broccoli. Of those who voted for other parties, 90% had never eaten broccoli. Of those who voted, 46% had eaten broccoli.
What percentage voted for the Broccoli Party?

13. A manager in a store has to determine the price of a sweater. Market research gives him the following data: If the price is €75, then 100 teenagers will buy the sweater. Each time the price is increased by €5, 20 fewer teenagers will buy the sweater. However, each time the price is decreased by €5, 20 sweaters more will be sold. The sweaters cost the company €30 apiece.
What is the sale price that maximizes profits?

14. The diagram shows a rectangle *ABCD* with $AB = 16$ and $BC = 12$. Angle *ACE* is a right angle and $CE = 15$. The line segments *AE* and *CD* meet at *F*.
What is the area of triangle ACF?



15. For each real number x, let $f(x)$ be the minimum of the numbers $3x + 1$, $2x + 3$ and $-4x + 24$.
What is the maximum value of $f(x)$?

16. The integer m has ninety-nine digits, all of them nines.
What is the sum of the digits of m^2 ?

17. In rectangle *ABCD*, the midpoints of sides *BC*, *CD* and *DA* are *P*, *Q* and *R* respectively. The point *M* is the midpoint of *QR*. The area of triangle *APM* is a fraction m / n of the area of rectangle *ABCD*, where m and n are integers and m / n is in its simplest form.
What is the value of $m + n$?

18. The integers a, b and c are such that $0 < a < b < c < 10$. The sum of all three-digit numbers that can be formed by a permutation of these three integers is 1554.
What is the value of c?

19. Given that $\left(a + \frac{1}{a}\right)^2 = 6$ and $a^3 + \frac{1}{a^3} = N\sqrt{6}$ and $a > 0$, what is the value of N?

20. The polynomial $f(x)$ is such that $f(x^2 + 1) \equiv x^4 + 4x^2$ and $f(x^2 - 1) \equiv ax^4 + 4bx^2 + c$.
What is the value of $a^2 + b^2 + c^2$?

SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

Friday 30th November 2012

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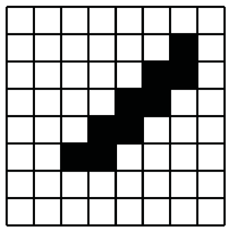
1. How many zeroes are there at the end of the number which is the product of the first 2012 prime numbers?

2. The size of the increase from each term to the next in the list a, $225\frac{1}{2}$, c, d, 284 is always the same. What is the value of a?

3. On the grid shown in the diagram, the shaded squares form a region, A.

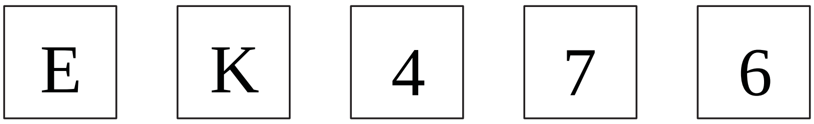


What is the maximum number of additional grid squares which can be shaded to form a region B such that B contains A and that the lengths of the perimeters of A and B are the same?

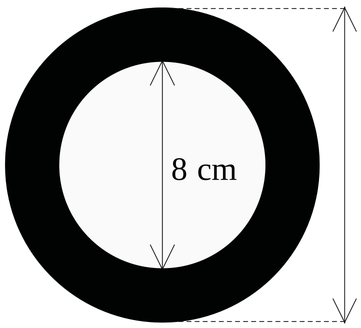
4. Five cards are laid on a table, as shown. Every card has a letter on one side and a number on the other side.

Peter says: “For every card on the table, if there is a vowel on one side of the card, then there is an even number on the other side.”

What is the smallest number of cards Sylvia must turn over in order to be certain that Peter is telling the truth?



5. Susan has two pendants made of the same material. They are equally thick and weigh the same. The first pendant is in the shape of an annulus created from two concentric circles, with diameters 8 cm and 12 cm, as shown. The shape of the second pendant is a disc. The diameter of the second pendant is written in the form $a\sqrt{b}$, where a is an integer and b is a prime integer.



What is the value of $a + b$?

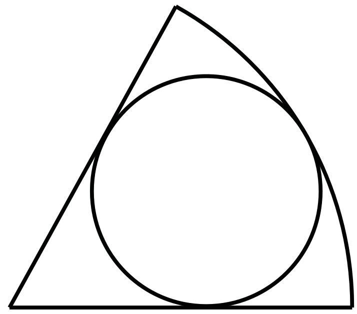
6. Given that $4^x = 9$ and $9^y = 256$, what is the value of xy?

7. When 1001 is divided by a single-digit number, the remainder is 5.
What is the remainder when 2012 is divided by the same single-digit number?

8. The three prime numbers a, b and c are such that $a > b > c$, $a + b + c = 52$ and $a - b - c = 22$.
What is the value of abc?

9. The diagram shows a circle touching a sector of another circle in three places. The ratio of the radius of the sector to the radius of the small circle is 3:1. The ratio of the area of the sector to the area of the small circle, written in its simplest form, is $p : q$.

What is the value of $p + q$?



10. Sixteen teams play in a volleyball league. Each team plays one game against every other team. For each game, the winning team is awarded 1 point, and the losing team 0 points. There are no draws. After all the games have been played and the teams have been ranked according to their total scores, the total scores form a sequence where the difference between consecutive terms is constant.

How many points did the team in first place receive?

11. Last year there were 30 more boys than girls in the school choir. This year the number of choir members has increased by 10%, the number of girls has increased by 20% and the number of boys by 5%.

How many members does the choir have this year?

12. The cells of a 4×4 grid are coloured black and white as shown in Figure 1. One move allows us to exchange the colourings of any two cells positioned in the same row or in the same column.

What is the minimum number of moves needed to obtain Figure 2?

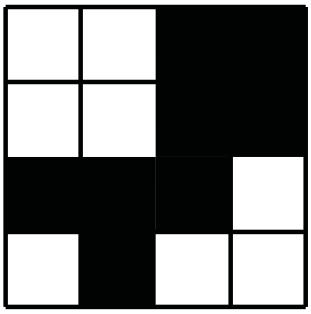


Figure 1

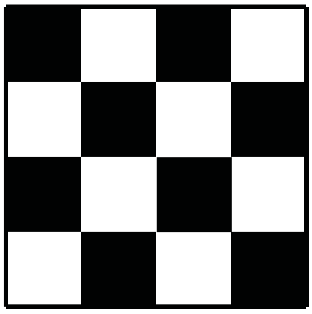
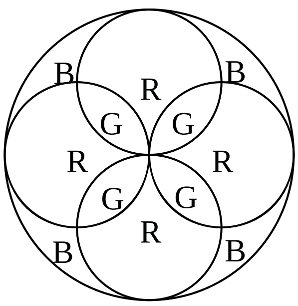


Figure 2

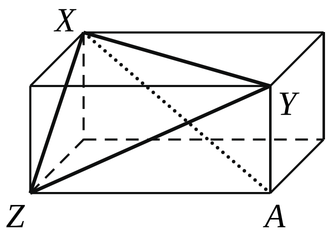
13. A circular stained-glass window is shown in the diagram. The four smaller circles are the same size and are positioned at equal intervals around the centre of the large circle. The letters R, G and B have been placed in regions of red, green and blue glass respectively. The total area of the green glass is 400.

What is the area of the blue glass?



14. The diagram shows a cuboid. In triangle XYZ, the lengths of XY, XZ and YZ are 9, 8 and $\sqrt{55}$ respectively.

What is the length of the diagonal XA shown?



15. The equation $x^2 - bx + 80 = 0$, where $b > 0$, has two integer-valued solutions. What is the sum of the possible values of b?

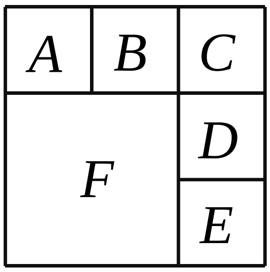
16. Given that $a + b = 5$ and $ab = 3$, what is the value of $a^4 + b^4$?

17. David removed one number from ten consecutive natural numbers. The sum of the remaining numbers was 2012.

Which number did he remove?

18. The diagram shows a square divided into six smaller squares labelled A, B, C, D, E and F. Two squares are considered to be adjacent if they have more than one point in common. The numbers 1, 2, 3, 4, 5 and 6 are to be placed in the smaller squares, one in each, so that no two adjacent squares contain numbers differing by 3.

How many different arrangements are possible?



19. A rectangle which has integer-length sides and area 36 is cut from a square with sides of length 20 so that one of the sides of the rectangle forms part of one of the sides of the square.

What is the largest possible perimeter of the remaining shape?

20. How many subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ exist in which the sum of the largest element and the smallest element is 11?

SENIOR ‘KANGAROO’ MATHEMATICAL CHALLENGE

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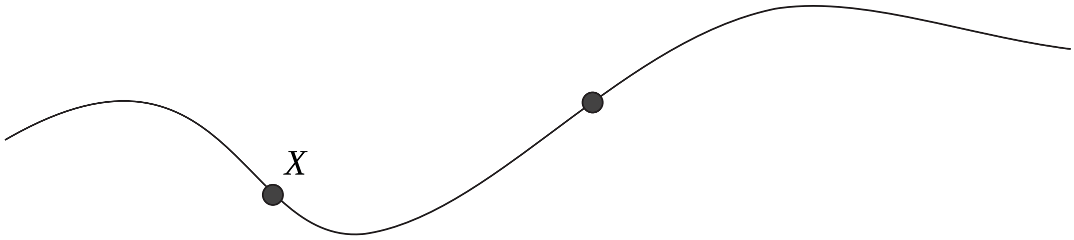
1. Adam, Bill and Carl have 30 sweets between them. Bill gives 5 sweets to Carl, Carl gives 4 sweets to Adam and Adam gives 2 sweets to Bill. Now each of them has the same number of sweets. How many sweets did Carl have initially?

2. An i-rectangle is defined to be a rectangle all of whose sides have integer length. Two i-rectangles are considered to be the same if they have the same side-lengths. The sum of the areas of all the different i-rectangles with perimeter 22 cm is A cm^2. What is the value of A?

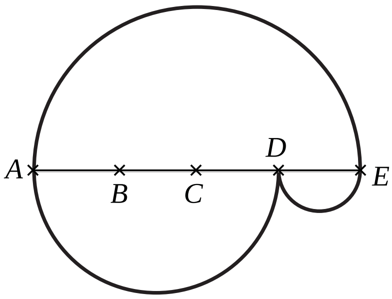
3. Some historians claim that the ancient Egyptians used a rope with two knots tied in it to construct a right-angled triangle by joining the two ends of the rope and taking the vertices of the triangle to be at the two knots and at the join. The length of the rope shown is 60 m and one of the knots is at X, which is 15 m from one end of the rope. How many metres from the other end of the rope should the second knot be placed to be able to create a right-angled triangle with the right angle at X?



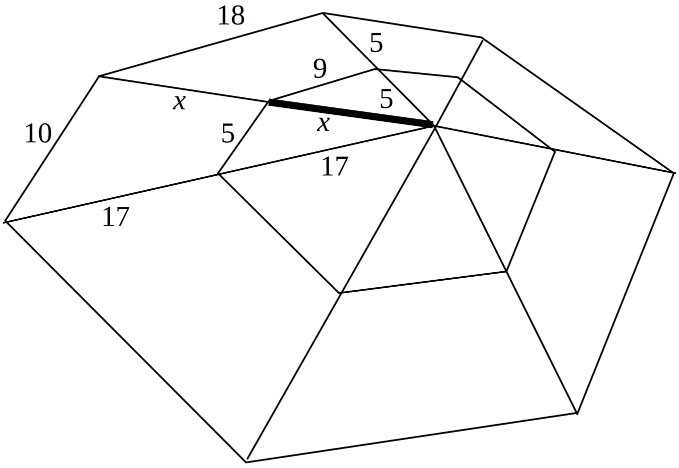
4. The height, width and length of a cube are multiplied by 2, 3 and 6 respectively to create a cuboid. The surface area of the cuboid is N times the surface area of the original cube. What is the value of N?

5. In a university admissions test, Dean gets exactly 10 of the first 15 questions correct. He then answers all the remaining questions correctly. Dean finds out he has answered 80% of all the questions correctly. How many questions are there on the test?

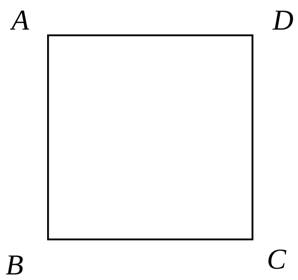
6. In the diagram, AE is divided into four equal parts and semicircles have been drawn with AE, AD and DE as diameters. This has created two new paths, an upper path and a lower path, from A to E. The ratio of the length of the upper path to the length of the lower path can be written as $a : b$ in its lowest terms. What is the value of $a + b$?



7. A mathematically skilful spider has spun a web and the lengths of some of the strands (which are all straight lines) are as shown in the diagram. It is known that x is an integer. What is the value of x?



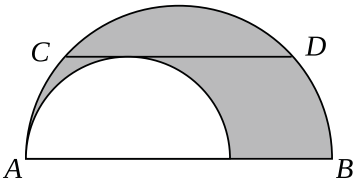
8. The square *ABCD* has sides of length 1. All possible squares that share two vertices with *ABCD* are drawn. The boundary of the region formed by the union of these squares is an irregular polygon. What is the area of this polygon?



9. In triangle *ABC*, angle *B* is 25% smaller than angle *C* and 50% larger than angle *A*. What is the size in degrees of angle *B*?

10. In the equation $2^{m+1} + 2^{m} = 3^{n+2} - 3^{n}$, m and n are integers. What is the value of m?

11. The diagram shows two semicircles. The chord *CD* of the larger semicircle is parallel to *AB*, and touches the smaller semicircle. The length of *CD* is 32 m. The area of the shaded region is $k\pi$ m^2. What is the value of k?



12. The sum of five consecutive integers is equal to the sum of the next three consecutive integers. What is the largest of these eight integers?

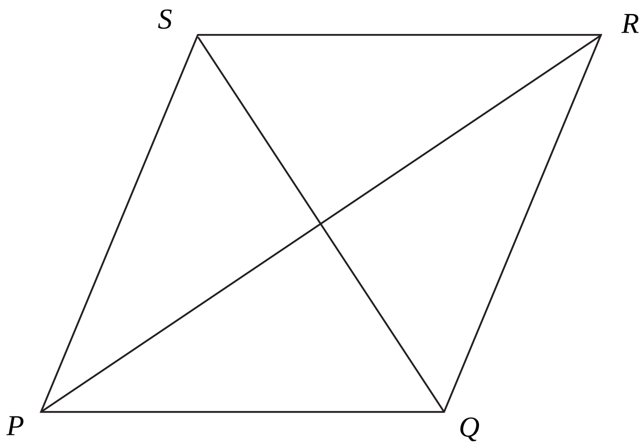
13. Zoe was born on her mother's 24th birthday so they share birthdays. Assuming they both live long lives, on how many birthdays will Zoe's age be a factor of her mother's age?

14. What is the largest three-digit integer that can be written in the form $n + \sqrt{n}$ where n is an integer?

15. How many integers a are there for which the roots of the quadratic equation $x^2 + ax + 2013 = 0$ are integers?

16. A sphere of radius 3 has its centre at the origin. How many points on the surface of the sphere have coordinates that are all integers?

17. The length of each side of the rhombus $PQRS$ is equal to the geometric mean of the lengths of its diagonals. What is the size in degrees of the obtuse angle PQR?



[The geometric mean of 2 values x_1 and x_2 is given by $\sqrt{x_1 x_2}$.]

18. How many of the first 2013 triangular numbers are multiples of 5?

19. The increasing sequence 1, 3, 4, 9, 10, 12, 13, … contains all the powers of 3 and all the numbers that can be written as the sum of two or more distinct powers of 3. What is the 70th number in the sequence?

20. Rachel and Nicky stand at either end of a straight track. They then run at constant (but different) speeds to the other end of the track, turn and run back to their original end at the same speed they ran before. On their first leg, they pass each other 20 m from one end of the track. When they are both on their return leg, they pass each other for a second time 10 m from the other end of the track. How many metres long is the track?

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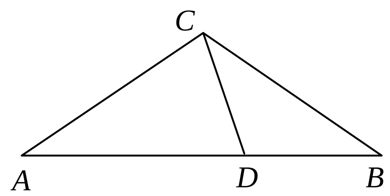
1. Three standard dice are stacked in a tower so that the numbers on each pair of touching faces add to 5. The number on the top of the tower is even. What is the number on the base of the tower?

2. How many prime numbers p have the property that $p^4 + 1$ is also prime?

3. Neil has a combination lock. He knows that the combination is a four-digit number with first digit 2 and fourth digit 8 and that the number is divisible by 9. How many different numbers with that property are there?

4. In the diagram, triangle ABC is isosceles with $CA = CB$ and point D lies on AB with $AD = AC$ and $DB = DC$. What is the size in degrees of angle BCA?

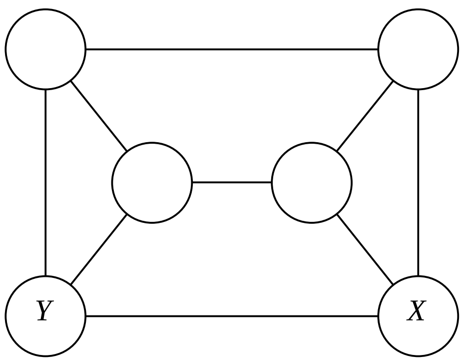


5. Six of the seven numbers 11, 20, 15, 25, 16, 19 and 17 are divided into three groups of two numbers so that the sum of the two numbers in each group is the same. Which number is not used?

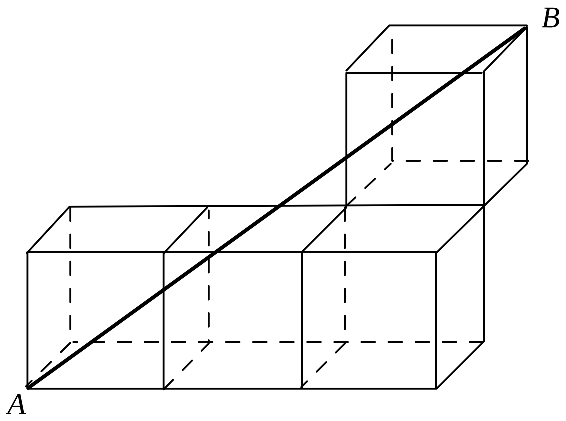
6. The numbers x, y and z satisfy the equations $x^2yz^3 = 7^3$ and $xy^2 = 7^9$. What is the value of $\frac{xyz}{7}$?

7. A table of numbers has 21 columns labelled 1, 2, 3, ..., 21 and 33 rows labelled 1, 2, 3, ..., 33. Every element of the table is equal to 2. All the rows whose label is not a multiple of 3 are erased. All the columns whose label is not an even number are erased. What is the sum of the numbers that remain in the table?

8. Andrew wishes to place a number in each circle in the diagram. The sum of the numbers in the circles of any closed loop of length three must be 30. The sum of the numbers in the circles of any closed loop of length four must be 40. He places the number 9 in the circle marked X. What number should he put in the circle marked Y?



9. Each of the cubes in the diagram has side length 3 cm. The length of *AB* is $\sqrt{k}$ cm. What is the value of *k*?

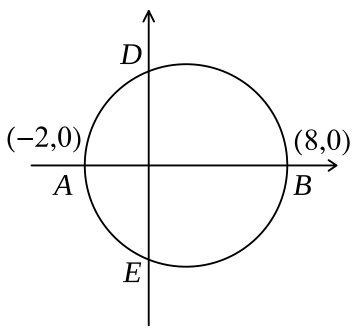


10. A Mathematical Challenge consists of five problems, each of which is worth a different whole number of marks. Carl solved all five problems correctly. He scored 10 marks for the two problems with the lowest numbers of marks and 18 marks for the two problems with the highest numbers of marks. How many marks did he score for all five problems?

11. The mean weight of five children is 45 kg. The mean weight of the lightest three children is 42 kg and the mean weight of the heaviest three children is 49 kg. What is the median weight of the children in kg?

12. On Old MacDonald's farm, the numbers of horses and cows are in the ratio 6:5, the numbers of pigs and sheep are in the ratio 4:3 and the numbers of cows and pigs are in the ratio 2:1. What is the smallest number of animals that can be on the farm?

13. The diagram shows a circle with diameter *AB*. The coordinates of *A* are (−2, 0) and the coordinates of *B* are (8, 0). The circle cuts the *y*-axis at points *D* and *E*. What is the length of *DE*?

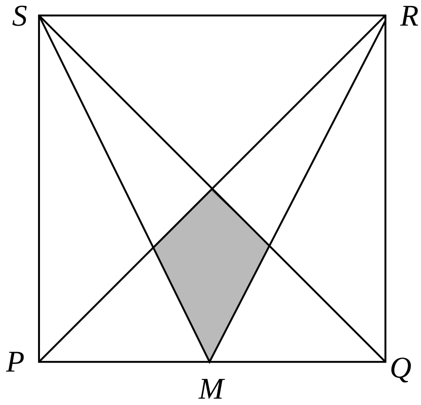


14. Rachel draws 36 kangaroos using three different colours. 25 of the kangaroos are drawn using some grey, 28 are drawn using some pink and 20 are drawn using some brown. Five of the kangaroos are drawn using all three colours. How many kangaroos did she draw that use only one colour?

15. A box contains seven cards numbered from 301 to 307. Graham picks three cards from the box and then Zoe picks two cards from the remainder. Graham looks at his cards and then says "I know that the sum of the numbers on your cards is even". What is the sum of the numbers on Graham's cards?

16. The numbers x, y and z satisfy the equations $x + y + z = 15$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$. What is the value of $x^2 + y^2 + z^2$?

17. In the diagram, *PQRS* is a square. *M* is the midpoint of *PQ*. The area of the square is *k* times the area of the shaded region. What is the value of *k*?



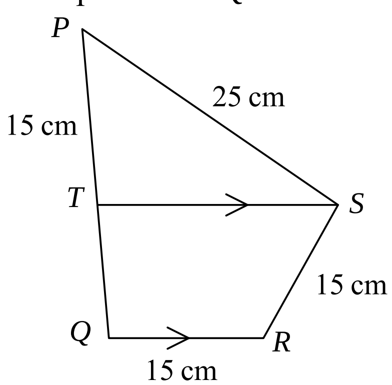
18. Twenty-five workmen have completed a fifth of a project in eight days. Their foreman then decides that the project must be completed in the next 20 days. What is the smallest number of additional workmen required to complete the project on time?

19. In the long multiplication sum shown, each asterisk stands for one digit.

$$\begin{array}{r} *** \\ \times \quad *** \\ \hline 22** \\ 90*0 \\ **2** \\ \hline 56*** \end{array}$$

What is the sum of the digits of the answer?

20. In the quadrilateral *PQRS* with $PQ = PS = 25$ cm and $QR = RS = 15$ cm, point *T* lies on *PQ* so that $PT = 15$ cm and so that *TS* is parallel to *QR*. What is the length in centimetres of *TS*?



SENIOR ‘KANGAROO’ MATHEMATICAL CHALLENGE

Friday 2nd December 2011

Organised by the United Kingdom Mathematics Trust

SOLUTIONS

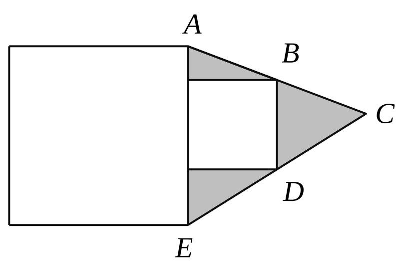
1. **71** Since the entries are different, no entry can be 1, and the smallest total will come from using the integers 2, 3 and 4 on the top line of the diagram. Treating an ordering and its reverse as the same, since they give the same total, we can arrange these in the orders 2, 3, 4 or 2, 4, 3 or 3, 2, 4. Of these, 3, 2, 4 gives the smallest overall total of 71.

2. **390** Over the years 2007 to 2010, the school accepted $325 \times 4 = 1300$ students. The mean for 2007 to 2011 is 4% higher than 325, which is 338. This means that $338 \times 5 = 1690$ students were accepted over the years 2007 to 2011. Therefore 390 students were accepted in 2011.

3. **16** On the first pass, all 200 people receive a pound coin. On the second pass, only people in even numbered positions receive a coin. On the *n*th pass, people receive a coin if *n* divides the number representing their position. Now 120 is divisible by 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60 and 120 so the 120th person receives 16 pound coins.

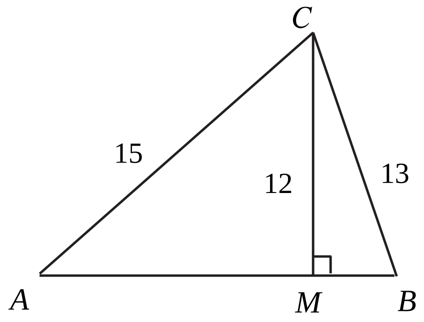
[*Note that, in general, if the prime decomposition of an integer, X, is* $p_1^{a_1}p_2^{a_2}p_3^{a_3}\ldots\ p_n^{a_n}$ *then the number of divisors of X is* $(a_1 + 1)(a_2 + 1)(a_3 + 1)\ldots\ (a_n + 1)$.]

4. **100** Using the labelling shown, we see that ΔACE and ΔBCD are similar and have lengths in the ratio 2:1. Because the height of ΔACE is 10 + the height of ΔBCD, the height of ΔACE is 20 and its area is $\frac{1}{2} \times 20 \times 20 = 200$. The area of the smaller square is 100 so the shaded area is $200 - 100 = 100$.



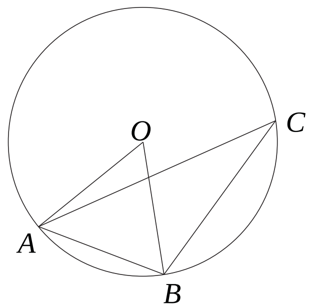
5. **36** The square and cube of an integer end in the same digit if, and only if, the integer itself ends in 0, 1, 5 or 6. The two-digit numbers with this property can have any tens digit from 1 to 9 so there are $4 \times 9 = 36$ such two-digit numbers.

6. **84** The perpendicular height from any side of an acute-angled triangle is always less than the length of either of the other two sides so the height is 12. Thus we have the situation shown in the diagram alongside.



Using Pythagoras' theorem in ΔACM, we obtain $AM = 9$ and in ΔCMB we get $MB = 5$. This means that $AB = 14$ and the area of triangle ABC is $\frac{1}{2} \times 14 \times 12 = 84$.

7. **30** Let the centre of the circle be O. Join A to C and O to A and B, as shown in the diagram alongside.



In ΔAOB, $AO = OB$ since they are both radii, and we are given that AB has length equal to the radius so $AB = AO = OB$ and ΔAOB is equilateral. Hence $\angle AOB = 60°$. Since the angle at the centre is twice the angle on the circumference, $\angle ACB$ is 30°.

8. **13** Let the original price, in pence, be p.
The new price is 4% more than the original so, working in pence,

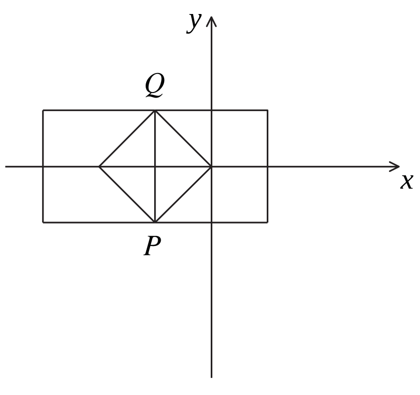
$$100n = \frac{104}{100} \times p,$$

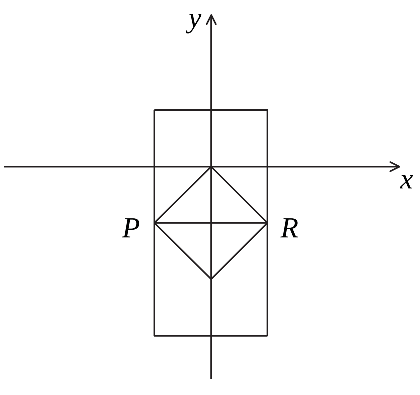
which may be rearranged to

$$n = \frac{13}{1250} \times p.$$

Now n is an integer and 1250 is not divisible by 13, so p is divisible by 1250. The smallest value of n will be when $p = 1250$, which means n is 13.

9. **5**





Let P be $(-1, -1)$ and suppose that the x-axis is a line of symmetry. Then $Q\,(-1, 1)$ is a vertex of the square since it is the reflection of the vertex P in the x-axis. Hence PQ is either an edge or a diagonal of the square. In the first case there are two possible squares and in the second case there is one, as shown in the first figure.

Similarly, when the y-axis is a line of symmetry there are three possible squares. However, one of these is the same as before, so in all there are exactly five squares possible.

10. **4** By expanding the brackets, we obtain

$$\left(\sqrt{8+2\sqrt{7}}-\sqrt{8-2\sqrt{7}}\right)^2 = 8+2\sqrt{7}-2\sqrt{\left(8+2\sqrt{7}\right)\left(8-2\sqrt{7}\right)}+8-2\sqrt{7}$$

$$= 16-2\sqrt{64-28} = 16-2\sqrt{36} = 16-12 = 4.$$

11. **60** Area $\triangle ABC$ = area $\triangle ACD$ + area $\triangle BCD$ + area $\triangle ABD$

$$= \tfrac{1}{2} \times e \times 5 + \tfrac{1}{2} \times f \times 12 + \tfrac{1}{2} \times g \times 13 = \tfrac{1}{2}\left(5e + 12f + 13g\right).$$

But $\triangle ABC$ has sides 5, 12 and 13, hence it is a right-angled triangle and so has area $\tfrac{1}{2} \times 5 \times 12 = 30$. Therefore $5e + 12f + 13g = 60$.

12. **40** From the information given about those who voted, we can conclude:

		Eaten broccoli? Yes	Eaten broccoli? No
Voted Broccoli Party?	Yes	y	0
	No	x	$9x$

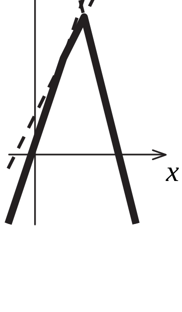
where x, $9x$ and y are the appropriate percentages of those who voted.

We are given that $x + y = 46$ and, since the table includes everyone, we also have $x + y + 9x = 100$. So $9x = 54$ and $x = 6$. Therefore $y = 40$ and so the percentage that voted for the Broccoli Party is 40%.

13. **65** Suppose n represents the number of increments of €5 above (or below, if n is negative) the selling price of €75. Then the number of sweaters sold is $100 - 20n$ and the profit made, in Euros, is $\left((75+5n)-30\right)(100-20n) = (45+5n)(100-20n) = 100(5-n)(9+n)$. So the profit is $100\left(49 - (n+2)^2\right)$ and is a maximum when $n = -2$. This gives a sale price of €65.

14. **75** Using Pythagoras' theorem firstly in $\triangle ABC$ and then in $\triangle ACE$ we get $AC = 20$ and $AE = 25$. It follows that $\triangle ABC$ is similar to $\triangle ACE$ as the corresponding sides are in the same ratio. Therefore, $\angle BAC = \angle CAE$. Also $\angle BAC = \angle ACF$, using alternate angles, so $\angle CAF = \angle ACF$ and $\triangle AFC$ is isosceles. Let M be the mid-point of AC and join M to F. This gives two more right-angled triangles, $\triangle AMF$ and $\triangle CMF$, also similar to $\triangle ABC$. Thus $\frac{MF}{MA} = \frac{BC}{BA}$ which gives $MF = \frac{15}{2}$. Therefore the area of $\triangle ACF$ is $\frac{1}{2} \times \frac{15}{2} \times 20 = 75$.

15. **10**

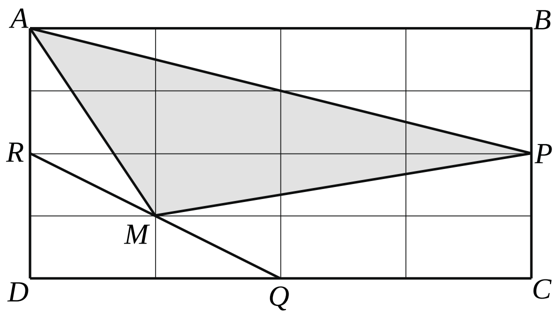


The diagram shows the dashed lines with equations $y = 3x + 1$, $y = 2x + 3$ and $y = -4x + 24$. The solid lines form the graph of the function given in the question. We can see that the maximum value of $f(x)$ occurs when $y = 2x + 3$ crosses $y = -4x + 24$.

At this point $y = 10$, therefore the maximum value of $f(x)$ is 10.

16. 891 Since $m = 10^{99} - 1$, we have $m^2 = (10^{99} - 1)^2 = 10^{198} - 2 \times 10^{99} + 1 = 999\ldots 9998000\ldots 001$ where there are 98 nines and 98 zeroes. Therefore the sum of the digits is $98 \times 9 + 8 + 1 = 891$.

17. 21



Dividing rectangle $ABCD$ into 16 equal parts, as shown in the diagram above, demonstrates that the area of $\Delta APM = 12 - \frac{1}{2} \times 3 - \frac{1}{2} \times 3 - \frac{1}{2} \times 8 = 5$ parts. Therefore the area of ΔAPM is $\frac{5}{16}$ of the area of rectangle $ABCD$ so $m + n = 21$.

18. 4 There are six different numbers that can be formed with digits a, b and c. The sum of these six numbers is

$$\begin{aligned} &(100a + 10b + c) + (100a + 10c + b) + (100b + 10a + c) \\ &+ (100b + 10c + a) + (100c + 10a + b) + (100c + 10b + a) \\ &= 200(a + b + c) + 20(a + b + c) + 2(a + b + c) \\ &= 222(a + b + c) = 1554 \end{aligned}$$

so $a + b + c = 7$. Thus the only possibility for a, b and c is 1, 2 and 4 so $c = 4$.

19. 3 From $\left(a + \frac{1}{a}\right)^2 = 6$ we have $a + \frac{1}{a} = \sqrt{6}$ since $a > 0$. Therefore $\left(a + \frac{1}{a}\right)^3 = (\sqrt{6})^3$, which gives $a^3 + 3a^2 \times \frac{1}{a} + 3a \times \frac{1}{a^2} + \frac{1}{a^3} = 6\sqrt{6}$ and so $N\sqrt{6} + 3\left(a + \frac{1}{a}\right) = 6\sqrt{6}$. This means that $N = 3$.

20. 17 From $f(x^2 + 1) \equiv x^4 + 4x^2 \equiv (x^2 + 1)^2 + 2(x^2 + 1) - 3$ we deduce that $f(w) \equiv w^2 + 2w - 3$ and hence that $f(x^2 - 1) \equiv (x^2 - 1)^2 + 2(x^2 - 1) - 3 \equiv x^4 - 2x^2 + 1 + 2x^2 - 2 - 3 \equiv x^4 - 4$. This means $a = 1$, $b = 0$ and $c = -4$. Therefore the value of $a^2 + b^2 + c^2$ is 17.

An alternative solution is to realise that $f(x^2 + 1) \equiv [(x^2 + 1) + 1]^2 - 4$. So $f(x^2 - 1) \equiv [(x^2 - 1) + 1]^2 - 4 \equiv x^4 - 4$. This gives the same value for $a^2 + b^2 + c^2$.

SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

Friday 30th November 2012

Organised by the United Kingdom Mathematics Trust

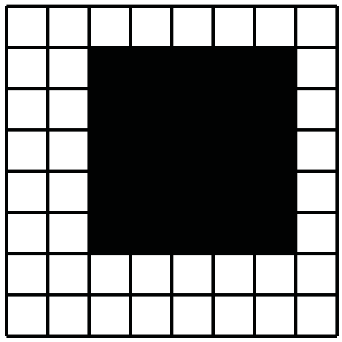
SOLUTIONS

2012 Senior Kangaroo Solutions

1. **1** In general, each zero at the end of an integer arises because, in the prime factorisation of the integer, there is a factor of 2 and a factor of 5 that can be paired to give a factor of 10. For example, $38\ 000 = 2^4 \times 5^3 \times 19$ so 2 and 5 may be paired three times giving a factor of 1000. The product of the first 2012 prime numbers only contains a single factor of 2 and a single factor of 5 so there is only one zero at the end.

2. **206** Let the increase from one term to the next be i. From $225\frac{1}{2}$ to 284 the increase is $3i$ so $284 - 225\frac{1}{2} = 3i$. Therefore $3i = 58\frac{1}{2}$ and hence $i = 19\frac{1}{2}$. To find the value of a, we need to decrease $225\frac{1}{2}$ by i which gives $a = 206$.

3. **16** The diagram shows a region R, say, that certainly contains the region A, and has the same perimeter as A. We claim that R is the region with the largest possible area with these properties and so is B, the region required. To see this, observe that adding any number of extra grid squares to R will only increase the perimeter and so will not give a region of the type required. Therefore, the maximum number of additional grid squares that can be added is 16.



4. **2** For Sylvia to be certain that Peter is telling the truth, she must check that the card showing the letter E has an even number on the other side and that the card showing the number 7 does not have a vowel on the other side. The other three cards do not need to be checked since K is not a vowel and cards with even numbers on one side may have any letter on the other side.

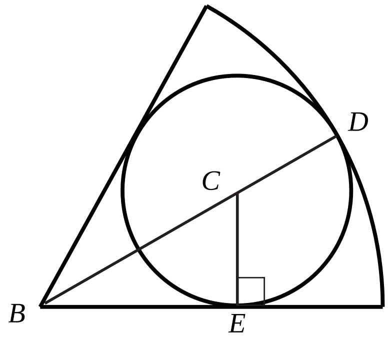
5. **9** For the weights of the two equally thick pendants to be the same, the area of the annulus must be equal to the area of the disc. The area of the annulus is $\pi \times 6^2 - \pi \times 4^2$ which is 20π. Since the area of a disc of radius r is πr^2, we have $\pi r^2 = 20\pi$ and so $r = \sqrt{20} = 2\sqrt{5}$. Thus the diameter of the second pendant is $2 \times 2\sqrt{5}$ which is $4\sqrt{5}$. So $a + b = 4 + 5 = 9$.

6. **4** We have $4^{xy} = (4^x)^y = 9^y = 256$. However $256 = 4^4$ so $xy = 4$.

7. **2** Let the single-digit number be k. Since the remainder is 5, k is larger than 5. Also k is a single-digit number so k is less than 10. Thus k is 6, 7, 8 or 9. However, the remainder when 1001 is divided by k is 5 so 996 is a multiple of k. Since 996 is not divisible by 7, 8 and 9 we conclude that $k = 6$. Finally, 2012 leaves a remainder of 2 when divided by 6.

8. **962** Adding the two given equations, we get $2a = 74$, which means $a = 37$. Substituting this value into the first equation we obtain $b + c = 15$. Therefore one of b and c is even and the other is odd. Since b and c are both prime, one of them is 2. This means the other is 13. Thus the product required is $2 \times 13 \times 37$, which is 962.

9. **5** Label the diagram as shown. Let the radius of the small circle be r, therefore $r = CD = CE$. We are told that the ratio of the radius of the sector to the radius of the small circle is 3 : 1 so $BD = 3 \times CD = 3r$. Therefore $BC = 2r$. Since triangle BCE is right-angled with two known sides, we have angle CBE is 30° and the area of the sector is a sixth of the area of a circle of radius $3r$. The ratio of the area of the sector to the area of the small circle is $\frac{1}{6}\pi(3r)^2 : \pi r^2$ which simplifies to $\frac{9}{6}\pi r^2 : \pi r^2$, this is, 3 : 2 in its simplest form. So $p + q = 5$.



10. **15** Each team played 15 matches so the maximum possible score for any one team is 15 points. Since the total score for a team must be an integer, the gap between consecutive team scores is also an integer. With 16 teams and no negative total scores, the gap is 1 point. This means the total scores are 15 points, 14 points, 13 points , …, 1 point, 0 points. Thus the team in first place scored 15 points, winning all their matches.

11. **99** Let the number of girls in the choir last year be x. This means there were $x + 30$ boys in the choir last year. This year there are 20% more girls, that is, $1.2x$ girls, and 5% more boys, that is, $1.05(x + 30)$ boys. The overall number in the choir this year is 10% more, that is, $1.1(x + x + 30)$. Putting these together, we get $1.1(2x + 30) = 1.2x + 1.05(x + 30)$. Multiplying out the brackets, we obtain $2.2x + 33 = 1.2x + 1.05x + 31.5$ and hence $15 = 0.05x$, so $x = 30$. The number in the choir this year is $2.2(2x + 30)$ which is therefore $2.2(2 \times 30 + 30)$. So there are 99 choir members this year.

12. **4** In the diagram, the squares labelled A, B, C, D, E and F all need to switch colour (from black to white or white to black, as appropriate). This could take as few as 3 exchanges, if the squares were distributed helpfully. Unfortunately, this is not the case. For example, each of A and F can only be paired with B. This means that it is impossible to use just three exchanges. So if we can perform the required switches in four exchanges, this must be the minimum number needed. This can be done in a number of ways, for example, exchange A and B, exchange C and D, exchange E and X, exchange X and F.

A			B
	C	D	
	E		X
			F

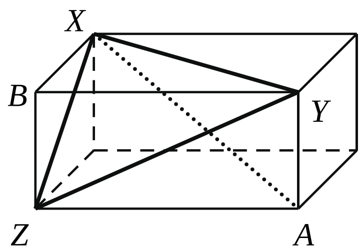
13. **400** Let the radius of each of the small circles be r. So the area of the whole window is $\pi(2r)^2$, which is $4\pi r^2$. Therefore $4\pi r^2 = 4(\mathrm{R} + \mathrm{G} + \mathrm{B})$ and $\pi r^2 = \mathrm{R} + \mathrm{G} + \mathrm{B}$. Now consider the area of one of the small circles. This is πr^2 but is also R + 2G. Equating these expressions for πr^2 gives R + G + B = R + 2G, which simplifies to B = G. Thus the area of the blue glass is equal to the area of the green glass.

Alternative: Note that the area of the large circle is equal to the area of four smaller circles since $\pi(2r)^2 = 4 \times \pi r^2$. But

the area of the window = the area of the 4 smaller circles − 4G + 4B.

Hence B = G.

14. **10** Let $BX = p$, $BY = q$ and $BZ = r$. Using Pythagoras' theorem in ΔBXY, ΔBXZ and ΔBYZ respectively, we obtain the equations: $p^2 + q^2 = 81$, $p^2 + r^2 = 64$ and $q^2 + r^2 = 55$. Adding all three equations, we get $2(p^2 + q^2 + r^2) = 81 + 64 + 55$ that is $p^2 + q^2 + r^2 = 100$. By applying Pythagoras' theorem to ΔBXZ and ΔAXZ, we find that the length of the diagonal XA is $\sqrt{p^2 + q^2 + r^2}$, which is $\sqrt{100} = 10$.



15. 186 The possible quadratic equations are $(x - 80)(x - 1) = 0$, $(x - 40)(x - 2) = 0$, $(x - 20)(x - 4) = 0$, $(x - 16)(x - 5) = 0$, $(x - 10)(x - 8) = 0$. These equations give the values of b as 81, 42, 24, 21 and 18 respectively. The sum of the possible values of b is 186.

16. 343 Expanding $(a + b)^4$ gives $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. We can rearrange this to give $a^4 + b^4 = (a + b)^4 - (4a^3b + 6a^2b^2 + 4ab^3)$ which can be written in the form $a^4 + b^4 = (a + b)^4 - (4ab(a^2 + b^2) + 6a^2b^2)$. Now using $a^2 + b^2 = (a + b)^2 - 2ab$ we can write $a^4 + b^4 = (a + b)^4 - (4ab((a + b)^2 - 2ab) + 6(ab)^2)$, that is $a^4 + b^4 = (a + b)^4 - 4ab(a + b)^2 + 2(ab)^2$. Substituting $a + b = 5$ and $ab = 3$, we obtain $a^4 + b^4 = 5^4 - (4 \times 3(5^2 - 2 \times 3) + 6 \times 3^2)$ which reduces to 343.

Note: For a quicker solution, observe that $a^2 + b^2 = (a + b)^2 - 2ab = 5^2 - 2 \times 3 = 19$ so that $a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2 = 19^2 - 2(ab)^2 = 361 - 18 = 343$.

17. 223 Let the smallest of the ten original numbers be x. The sum of all ten numbers is $x + (x + 1) + (x + 2) + \ldots + (x + 9)$, which equals $10x + 45$.

Let the number removed be $x + a$ where $0 \leqslant a \leqslant 9$. Removing this number from the ten original numbers leaves 2012 so $10x + 45 - (x + a) = 2012$, that is, $9x = 1967 + a$. Dividing by 9, we get $x = 218\frac{5}{9} + \frac{a}{9}$. Since x is an integer, $a = 4$. Hence $x = 219$ and the number removed, $x + a$, is 223.

18. 96 The numbers 1, 2, 3, 4, 5 and 6 can be paired uniquely (1 with 4, 2 with 5, and 3 with 6) so that the difference between the numbers in each pair is 3. Any of the six numbers can be placed in position F but once that has been done, the partner of this number must be placed in position C as, otherwise, the square containing F would be adjacent to a square containing a number that differs by 3. Any of the remaining four numbers may be placed at position A, but then the number placed at B must be one of the two numbers from the pair that has so far been left unused. Finally, the two unused numbers may be positioned at D and E, in either order. Thus the number of possible arrangements is $6 \times 4 \times 2 \times 2$, which is 96.

19. 116 Let the lengths of the sides of the rectangle be a and b, where $a < b$. For the rectangle to be cut from the square, $b < 20$. But $ab = 36$ and a and b are integers, so the greatest possible value of b is 18. Note that cutting the rectangle so that it shares a corner with the original square and so two sides of the rectangle form part of two sides of the original square would still leave the remaining shape with a perimeter of 80. Cutting the rectangle from the square with one side of length a taken to be part of one of the sides of the square will give a perimeter of $80 + 2b$. Similarly, if the side of length b is taken to be part of one of the sides of the square, the perimeter of the new shape will be $80 + 2a$. For the largest perimeter, it is clearly better to do the former since a is less than b. Now $80 + 2b$ is largest when b is largest, so putting $b = 18$ we obtain the largest perimeter which is 116.

20. 341 If the smallest element was 1 then the largest element would be 10 (to give a sum of 11). Of the remaining possible elements, 2 could be included or not, 3 could be included or not and so on. This means that there are 2^8 possible subsets with smallest element 1 and largest element 10. Similarly there are 2^6 possible subsets with smallest element 2 and largest element 9, there are 2^4 possible subsets with smallest element 3 and largest element 8, there are 2^2 possible subsets with smallest element 4 and largest element 7, and there is one possible subset with smallest element 5 and largest element 6. In total this means there are $2^8 + 2^6 + 2^4 + 2^2 + 1 = 341$ possible subsets.

SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

Friday 29th November 2013

Organised by the United Kingdom Mathematics Trust

SOLUTIONS

2013 Senior Kangaroo Solutions

1. **9** There are 30 sweets in total so, since the boys all finish with the same number of sweets, they must then have $30 \div 3 = 10$ sweets. Carl gains 5 sweets from Bill and gives 4 sweets to Adam so has a net gain of 1 sweet. Since Carl finishes with 10 sweets, he must start with $10 - 1 = 9$ sweets.

2. **110** The perimeter of each i-rectangle is 22 cm. Therefore, the sum of the length and the width is 11 cm. All the sides of the i-rectangle are whole numbers so the possible i-rectangles are 1×10 with area 10 cm^2, 2×9 with area 18 cm^2, 3×8 with area 24 cm^2, 4×7 with area 28 cm^2 and 5×6 with area 30 cm^2. Hence the sum of the areas of all possible i-rectangles is $10 + 18 + 24 + 28 + 30 = 110$ cm^2. Therefore the value of A is 110.

3. **25** Let the distance of the second knot from the other end of the rope be d m.

This part of the rope will become the hypotenuse of the right-angled triangle so, on applying Pythagoras' Theorem, we have the equation $d^2 = 15^2 + (45 - d)^2$. Now expand the brackets to get $d^2 = 225 + 2025 - 90d + d^2$. This simplifies to $90d = 2250$, which has solution $d = 25$.

Hence the second knot is 25 m from the other end of the rope.

4. **12** Let each side of the original cube have length x so that the cube has surface area $6x^2$. Then the cuboid has side-lengths $2x$, $3x$ and $6x$, so has surface area
$2 \times (2x \times 3x + 2x \times 6x + 3x \times 6x) = 72x^2$.

Hence the value of N is $72x^2 \div 6x^2 = 12$.

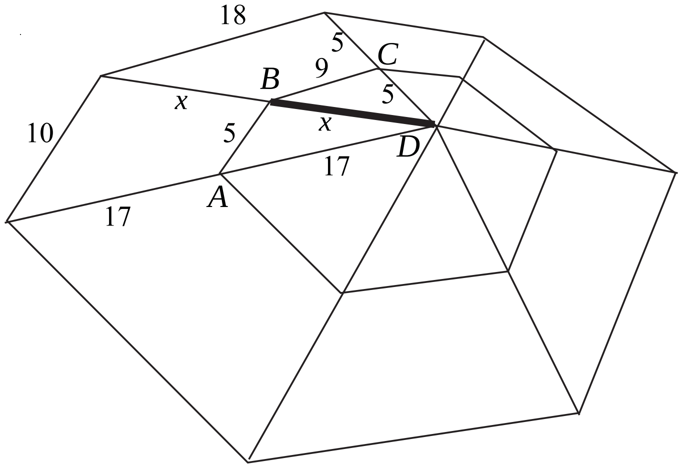
5. **25** Dean has answered 5 questions incorrectly so 5 questions must represent 20% of the questions. 20% is equivalent to $\frac{1}{5}$ so the total number of questions is $5 \times 5 = 25$.

6. **2** Let the length of AE be $4x$. Therefore, the lengths of AD and DE are $3x$ and x respectively.
The length of the upper path is $\frac{1}{2} \times \pi \times 4x = 2\pi x$.
The length of the lower path is $\frac{1}{2} \times \pi \times 3x + \frac{1}{2} \times \pi \times x = 2\pi x$.

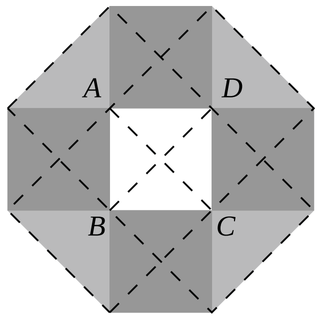
Therefore the ratio of the length of the upper path to the length of the lower path is 1 : 1. Hence the value of $a + b$ is 2.

7. **13** In any triangle, the length of the longest side is less than the sum of the lengths of the other two sides.

Apply this result (known as the triangle inequality) to the triangle BCD to obtain $x < 9 + 5$ or $x < 14$. In the same way, apply this result to triangle ABD to obtain $17 < x + 5$ or $x > 12$. But we are told that x is an integer and so $x = 13$.



8. **7** In addition to the original square, four squares can be drawn that share two adjacent vertices of the original square and a further four squares can be drawn that share two opposite vertices of the original square. The union of these squares creates the octagon as shown.



The octagon is made up of five squares of side 1 unit and four halves of a square of side 1 unit. Hence the area of the polygon is equal to $7 \times 1 \times 1 = 7$.

9. **60** Let the sizes of angles A, B and C be $a°$, $b°$ and $c°$ respectively.

From the question we have $b = 0.75 \times c$. Therefore $c = \frac{4}{3}b$.

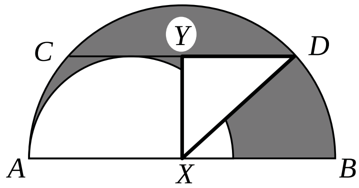
Similarly we have $b = 1.5 \times a$. Therefore $a = \frac{2}{3}b$.

Angles in a triangle add up to 180°, so that we have$180 = \frac{2}{3}b + b + \frac{4}{3}b$, which means that $180 = 3b$. It follows that $b = 60$.

10. **3** Factorise both sides of the equation to get $2^m(2^1 + 1) = 3^n(3^2 - 1)$. Thus we have $2^m \times 3 = 3^n \times 8$ which is equivalent to $2^{m-3} = 3^{n-1}$. Since 2 and 3 have no factors in common other than 1, a power of 2 cannot equal a power of 3 unless both powers are zero when both sides of the equation equal 1. Therefore we have $m - 3 = 0$ and $n - 1 = 0$. Hence the value of m is 3 (and the value of n is 1).

11. **128** Let the radii of the larger and smaller semicircles be R and r respectively. Then the shaded area as $\frac{1}{2} \times \pi R^2 - \frac{1}{2} \times \pi r^2 = \frac{1}{2}\pi(R^2 - r^2)$.

Let X be the centre of the larger semicircle and let Y be the midpoint of CD. Since CD is parallel to AB then $XY = r$ and $\angle XYD = 90°$. Apply Pythagoras' Theorem to triangle XYD to give $R^2 = r^2 + 16^2$ or $R^2 - r^2 = 256$. Therefore the shaded area is $\frac{1}{2}\pi(R^2 - r^2) = 128\pi$. Hence the value of k is 128.



12. **11** Let the smallest number be n. From the information in the question, we obtain the equation

$$n + n + 1 + n + 2 + n + 3 + n + 4 = n + 5 + n + 6 + n + 7.$$

Therefore we get $5n + 10 = 3n + 18$, which has solution $n = 4$.

Hence the largest number is $4 + 7 = 11$.

13. **8** Let Zoe be x years old. Therefore, her mother's age is $x + 24$ years old. Now x divides $x + 24$ if and only if x divides 24. The positive factors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24 and so Zoe's age is a factor of her mother's age on 8 birthdays, when her mother's age will be 25, 26, 27, 28, 30, 32, 36 and 48.

14. **992** Since $n + \sqrt{n}$ is an integer, n is a square number. The square numbers near 1000 are $30^2 = 900$, $31^2 = 961$ and $32^2 = 1024$. Clearly, if $n = 32^2$ then $n + \sqrt{n}$ is greater than 1000, so this is not possible. However, if $n = 31^2$, then $n + \sqrt{n} = 961 + 31 = 992$, which is less than 1000. Hence the largest three-digit integer than can be written in the given form is 992.

15. **8** If the equation $x^2 + ax + 2013 = 0$ has integer solutions, then it can be written in the form $(x + b)(x + c) = 0$ for integers b and c. This means that $bc = 2013$.

As the prime factorisation of 2013 is $3 \times 11 \times 61$, so the possible factor pairs of 2013 are 1 and 2013, 3 and 671, 11 and 183 and 33 and 61. However, these only take into account the cases when both b and c are positive and four further pairs are possible if both b and c are negative. Thus there are 8 distinct values of a, namely ±2014, ±674, ±194 and ±94.

16. **30** Let the coordinates of a relevant point on the sphere be (x, y, z). By the three-dimensional version of Pythagoras' Theorem, we have $x^2 + y^2 + z^2 = 3^2$. The only solutions for which x, y and z are positive integers are (3, 0, 0) and (1, 2, 2) in some order.

There are $3 \times 2 = 6$ solutions based on the values (3, 0, 0) as the 3 can go in any of the three positions and be either positive or negative. Similarly there are $3 \times 2 \times 2 \times 2 = 24$ solutions based on the values (1, 2, 2) as the 1 can go in any of the three positions and all three of the values can independently be either positive or negative. This gives $6 + 24 = 30$ solutions in total.

17. **150** Let x be the length of a side of the rhombus and let a and b be the lengths of the two diagonals. The area of the rhombus is

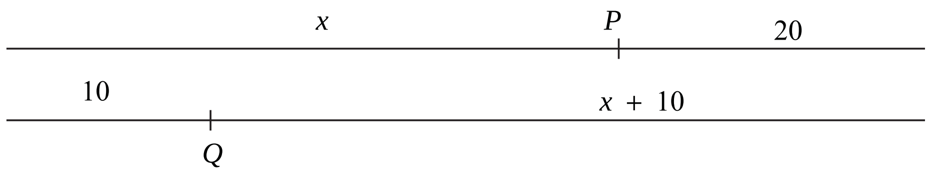
$$\text{area}\,\Delta QRS + \text{area}\,\Delta PQS = \tfrac{1}{2}x^2 \sin\angle SRQ + \tfrac{1}{2}x^2 \sin\angle SPQ.$$

Opposite angles in a rhombus are equal so this simplifies to $x^2 \sin\angle SRQ$. However, the area of a rhombus can also be calculated in a similar way to a kite, i.e. half the product of the diagonals. This gives the equation $x^2 \sin\angle SRQ = \tfrac{1}{2}ab$. From the question, we know that $x = \sqrt{ab}$ so $x^2 = ab$. Hence $\sin\angle SRQ = \tfrac{1}{2}$ and so $\angle SRQ = 30°$. Lines SR and PQ are parallel and so, using co-interior angles, $\angle PQR + \angle SRQ = 180°$. This means $\angle PQR = 150°$.

18. **804** The triangular numbers are given by the formula $T_n = \tfrac{1}{2}n(n + 1)$. T_n is a multiple of 5 if, and only if, one of n or $n + 1$ is also a multiple of 5. This means that two triangular numbers in every group of 5 consecutive triangular numbers will be a multiple of 5. None of the numbers 2011, 2012, 2013 or 2014 is a multiple of 5 and so none of T_{2011}, T_{2012} or T_{2013} is a multiple of 5 either. Hence the number of multiples of 5 in the first 2013 triangular numbers is $2 \times \frac{2010}{5} = 804$.

19. **741** Consider the n numbers $3^0, 3^1, 3^2, \ldots, 3^{n-1}$. Using at most one of each of these in a sum, the number of totals we can create is $2^n - 1$ as each of the n numbers can either be included or excluded from the sum but at least one number must be included so the choice of excluding all the numbers is discounted (and they are all distinct). For $n = 6$, this is 63 and so the 64th number in the sequence will be $3^6 = 729$. Then the 70th term is equal to the 64th term + 6th term = $729 + 12 = 741$.

20. **50** Let the distance Rachel runs before they first meet be x m. Let v_R and v_N be the respective speeds of Rachel and Nicky and let t_1 and t_2 be the times they take to get to their first and second passing points respectively (shown as P and Q on the diagram below).



As distance = speed × time, we have the following equations:

$$x = v_R t_1 \text{ and } 20 = v_N t_1;\ 2x + 30 = v_R t_2 \text{ and } x + 30 = v_N t_2$$

where the first and second pair give the distances travelled by Rachel and Nicky from the start to P and Q respectively.

Divide each equation in the first set by the corresponding equation in the second set to eliminate v_R and v_N to obtain $\dfrac{x}{2x + 30} = \dfrac{t_1}{t_2} = \dfrac{20}{x + 30}$. Now multiply both sides by the common denominator $(x + 30)(2x + 30)$ to obtain $x(x + 30) = 20(2x + 30)$. This simplifies to $x^2 + 30x = 40x + 600$, i.e. $x^2 - 10x - 600 = 0$. This factorises to $(x - 30)(x + 20) = 0$ with solutions $x = 30$ and $x = -20$. As x is measuring a distance, it must be positive so $x = 30$. Hence the length of the track is $30 + 20 = 50$ m.

SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

Friday 28th November 2014

Organised by the United Kingdom Mathematics Trust

SOLUTIONS

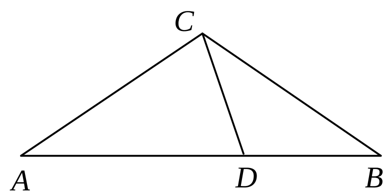
2014 Senior Kangaroo Solutions

1. **5** First note that, on a standard die, the numbers on opposite faces add to 7. Let the number on the top of the tower be n. The numbers on the touching faces going down the tower are then $7 - n$, $5 - (7 - n) = n - 2$, $7 - (n - 2) = 9 - n$ and $5 - (9 - n) = n - 4$ respectively. The bottom number is $7 - (n - 4) = 11 - n$. The numbers on a die are 1 to 6 so $11 - n \leqslant 6$ and hence $n \geqslant 5$. The question states that n is even so $n = 6$. Hence the number on the bottom of the tower is $11 - n = 5$. (It is easy to check that when $n = 6$ all the numbers going down the tower are values that can appear on the face of a standard die.)

2. **1** All prime numbers p greater than 2 are odd. For these numbers, $p^4 + 1$ is even and greater than 2 and so not prime. However, $2^4 + 1 = 17$ which is prime. Hence only one prime number, namely 2, has the desired property.

3. **11** A number is divisible by 9 if and only if the sum of its digits is divisible by 9. Let the second and third digits of the combination be x and y respectively. Hence $10 + x + y$ is divisible by 9. Since $0 \leqslant x \leqslant 9$ and $0 \leqslant y \leqslant 9$ we have $10 + x + y = 18$ or 27. This gives either $x + y = 8$, which has nine different solutions given by $x = 0$, $x = 1$, and so on up to $x = 8$ or $x + y = 17$ which has two different solutions, namely $x = 8$, $y = 9$ and $x = 9$, $y = 8$. This means there are $9 + 2 = 11$ different combinations with the desired property.

4. **108** Let $\angle CAB = x°$. Triangle ABC is isosceles with $CA = CB$ so $\angle CBA = x°$. Triangle BCD is also isosceles with $DB = DC$ so $\angle BCD = x°$.



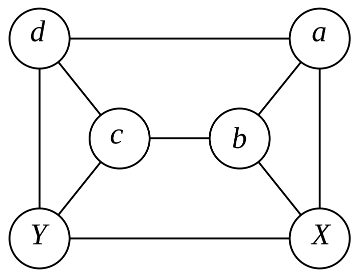
The exterior angle of any triangle is equal to the sum of the interior opposite angles, so $\angle CDA = 2x°$ and hence, since triangle CAD is isosceles, $\angle ACD = 2x°$.
The angle sum of a triangle is 180°, and applying this to triangle CAD we have $x + 2x + 2x = 180$. Therefore $x = 36$ and hence $\angle BCA = 36° + 2 \times 36° = 108°$.

5. **15** Let the number not used be x. The sum of the seven numbers is 123 which is divisible by 3. The six numbers used are divided into three pairs with the same sum so $123 - x$ is also divisible by 3. This means that x is divisible by 3 and the only number in the list that is divisible by 3 is 15. The remaining six numbers can then be paired as 11 and 25, 20 and 16, 19 and 17 all with sum 36.

6. **343** Multiply the two given equations together to obtain $x^2yz^3 \times xy^2 = 7^3 \times 7^9$. Hence $x^3y^3z^3 = 7^{12}$ and so $xyz = 7^4$. Therefore the value of $\frac{xyz}{7}$ is $7^3 = 343$.

7. **220** When the unwanted rows and columns are erased, 11 rows and 10 columns remain. The table then contains 11×10 entries, all equal to 2. Hence the sum of the numbers remaining in the table is $11 \times 10 \times 2 = 220$.

8. **11** Let the numbers placed in the empty circles be a, b, c and d as shown and let y be the number placed in the circle marked Y. Recall that the number placed in the circle marked X is 9. The sum of the numbers in a closed loop of length 3 is 30 so $a + b + 9 = 30$ and $c + d + y = 30$. Add these two equations to get $a + b + 9 + c + d + y = 60$. However, the sum of the numbers in a closed loop of length 4 is 40. Thus we also have $a + b + c + d = 40$. This tells us that $9 + y = 20$ and hence that $y = 11$ so Andrew should place number 11 in the circle marked Y.



9. 153 Use the three-dimensional version of Pythagoras' Theorem to get $AB^2 = (3 \times 3)^2 + (2 \times 3)^2 + (2 \times 3)^2 = 81 + 36 + 36$. Hence $AB^2 = 153$ so $k = 153$.

10. 35 Let the number of marks scored for each question be a, b, c, d and e with $a < b < c < d < e$. The number of marks scored for the two questions with the lowest number of marks is 10 and so $a + b = 10$. However, $a < b$ and so $b \geqslant 6$. Similarly $d + e = 18$ and $d < e$ and hence $d \leqslant 8$. So $6 \leqslant b < c < d \leqslant 8$ and therefore $b = 6$, $c = 7$ and $d = 8$. So the total number of marks Carl scored is $10 + 7 + 18 = 35$.

11. 48 The total weight of the five children is 5×45 kg = 225 kg. Similarly, the total weight of the three lightest children is 3×42 kg = 126 kg and the total weight of the three heaviest children is $3 \times 49\text{kg} = 147\text{kg}$. Since there are five children, the child with the median weight is both the third lightest and the third heaviest and so has been included in both of these groups. Hence the median weight is $126\text{kg} + 147\text{kg} - 225\text{kg} = 48\text{kg}$.

12. 123 Since the ratio of the numbers of horses to cows is 6 : 5, the number of cows must be a multiple of 5. Since the ratio of the numbers of cows to pigs is 2 : 1, the number of pigs must also be a multiple of 5. Also, since the ratio of the numbers of pigs to sheep is 4 : 3, the number of pigs is a multiple of 4. Hence the number of pigs is a multiple of 20. The smallest multiple of 20 is 20 itself and one can check that 20 pigs is feasible, with the numbers of horses, cows and sheep being 48, 40 and 15 respectively. This gives the smallest number of animals on the farm as 123.

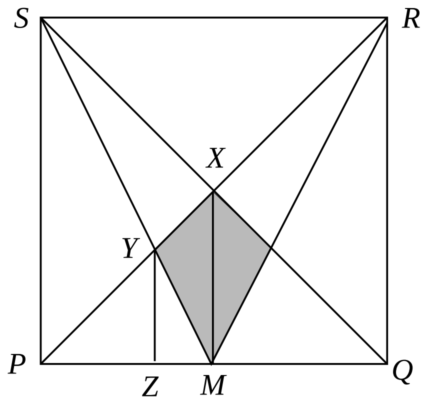
13. 8 The diameter of the circle is $8 - (-2) = 10$ units so the radius is 5 units. The centre of the circle is at X, the midpoint of AB, with coordinates (3, 0). Consider triangle OXD where O is the origin. This is a right-angled triangle with one side 3 units and hypotenuse 5 units so has third side 4 units. Thus the coordinates of D are (0, 4) and the coordinates of E will be (0, −4) by symmetry. Hence the length of DE is $4 - (-4) = 8$ units.

14. 4 Each kangaroo is drawn using one, two or three colours. So, for example, the number 25 of kangaroos drawn using some grey includes the kangaroos that are only grey, those that are grey and exactly one other colour and those that are all three colours. Therefore, by adding 25, 28 and 20 we count those kangaroos with just one colour once, we count those that have exactly two colours twice and those that have all three colours three times. Hence $25 + 28 + 20 = 36$ + (number with exactly two colours) + 2 × (number with three colours) which simplifies to $73 = 36$ + (number with exactly two colours) + 2×5. Hence the number drawn with exactly two colours is 27 and so the number drawn with only one colour is $36 - 27 - 5 = 4$.

15. 912 To be certain that the sum of the numbers on Zoe's two cards is even, the four cards that she chose from cannot contain cards of different parity (that is, they are all odd or all even). The original set of seven cards contained four odd-numbered cards and three even-numbered cards, so the only way a set of four cards all with the same parity can remain is if Graham chose the three even-numbered cards. Hence the sum of the numbers on Graham's cards is 302 + 304 + 306 = 912.

16. 225 Multiply each term of the second equation by xyz to obtain $yz + xz + xy = 0$. Square each side of the first equation to obtain $(x + y + z)^2 = 15^2$. So $x^2 + 2xy + 2xz + y^2 + 2yz + z^2 = 225$ and hence $x^2 + y^2 + z^2 = 225$.

17. 12 Let the length of the sides of the square be 2 units so its area is 4 units^2. Introduce points X, Y and Z as shown on the diagram where XM and YZ are parallel to SP and let the length of PZ be x units. The triangles PZY, PMX and PQR are all similar and isosceles so $YZ = x$ and $XM = 1$. Also triangles SPM and YZM are similar so $\frac{2}{1} = \frac{x}{1 - x}$ which has solution $x = \frac{2}{3}$. The shaded area is then $2 \times \frac{1}{2} \times 1 \times \left(1 - \frac{2}{3}\right) = \frac{1}{3}$. Hence the area of the square is $4 \div \frac{1}{3} = 12$ times the shaded area and so $k = 12$.



18. 15 The total number of 'man-days' of work required for the project is $5 \times 25 \times 8 = 1000$. The number of 'man-days' completed is $25 \times 8 = 200$ leaving 800 to be completed. To finish this in 20 days requires $800 \div 20 = 40$ workmen and so an extra $40 - 25 = 15$ workmen are required.

19. 16

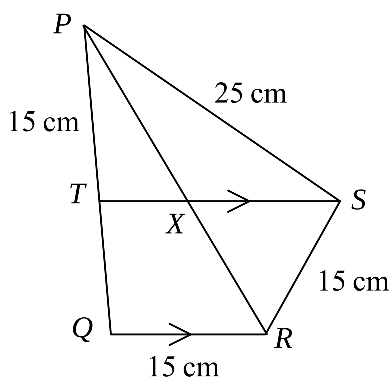
```
     ***
   × ***
   -----
    22**
    90*0
   **2**
   -----
   56***
```

The figures 2, 0 and 2 in the hundreds column lines 3, 4 and 5 of the calculation are not large enough to create any carry into the thousands column. Hence the first two missing figures in the third row of working must add with 2 and 9 to give 56 and so are 4 and 5. Note also that the final two digits in that row must be zeros from the structure of the sum so the third row of working is 45200. This means that one of the original 3-digit multiplicands is a 3-digit factor of 452 and so is 452, 226 or 113. The first row of the working is a 4-digit number starting 22 and so is 2260 as it is also a multiple of the same multiplicand. This means that this multiplicand is 452 and that the completed sum is as shown on the right.

```
     452
   × 125
   -----
    2260
    9040
   45200
   -----
   56500
```

Hence the sum of the digits of the answer is $5 + 6 + 5 + 0 + 0 = 16$.

20. 24 Draw in line PR as shown and let X be the point where PR intersects TS. The corresponding sides of ΔPQR and ΔPSR are equal and so ΔPQR and ΔPSR are congruent (SSS). Hence $\angle PRQ = \angle PRS$. Note also that, because TS and QR are parallel, $\angle PRQ = \angle RXS$ since they are alternate angles. This means that $\angle PRS = \angle RXS$ and so ΔXRS is isosceles and hence $XS = RS = 15$ cm.



TX is parallel to QR and so $\angle PTX = \angle PQR$ and $\angle PXT = \angle PRQ$ using corresponding angles. This means that ΔPTX and ΔPQR are similar and so $TX : QR = PT : PQ$ which gives $TX : 15 = 15 : 25$ so $TX = 9$ cm.

Hence $TS = TX + XS = 9\text{ cm} + 15\text{ cm} = 24$ cm.