Junior Mathematical Olympiad

Past Papers and Solutions 2010-2014

UKMT

United Kingdom Mathematics Trust

UK Junior Mathematical Olympiad 2010

Organised by The United Kingdom Mathematics Trust

Tuesday 15th June 2010

RULES AND GUIDELINES : READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING

1. Time allowed: 2 hours.
2. **The use of calculators, measuring instruments and squared paper is forbidden.**
3. All candidates must be in *School Year 8 or below* (England and Wales), *S2 or below* (Scotland), *School Year 9 or below* (Northern Ireland).
4. For questions in Section A *only the answer is required*. Enter each answer neatly in the relevant box on the Front Sheet. Do not hand in rough work. Write in blue or black pen or pencil.

   For questions in Section B you must give *full written solutions*, including clear mathematical explanations as to why your method is correct.

   Solutions must be written neatly on A4 paper. Sheets must be STAPLED together in the top left corner with the Front Sheet on top.

   Do not hand in rough work.
5. Questions A1-A10 are relatively short questions. Try to complete Section A within the first 45 minutes so as to allow well over an hour for Section B.
6. Questions B1-B6 are longer questions requiring *full written solutions*.
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Section A

A1 What is the value of $\frac{1}{1} + \frac{2}{\frac{1}{2}} + \frac{3}{\frac{1}{3}} + \frac{4}{\frac{1}{4}} + \frac{5}{\frac{1}{5}}$?

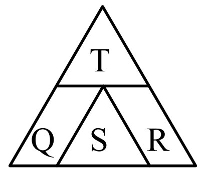
A2 Given that $x : y = 1 : 2$ and $y : z = 3 : 4$, what is $x : z$?

A3 Tom correctly works out 20^{10} and writes down his answer in full.
How many digits does he write down in his full answer?

A4 Three monkeys Barry, Harry and Larry met for tea in their favourite café, taking off their hats as they arrived. When they left, they each put on one of the hats at random. What is the probability that none of them left wearing the same hat as when they arrived?

A5 The sum of two positive integers is 97 and their difference is 37. What is their product?

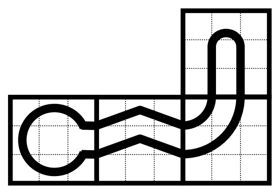
A6 In the diagram, the equilateral triangle is divided into two identical equilateral triangles S and T, and two parallelograms Q and R which are mirror images of each other.



What is the ratio of area R : area T ?

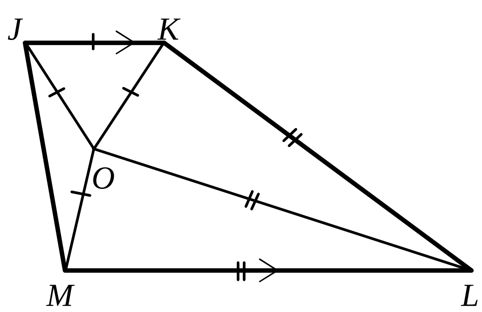
A7 What is the largest possible angle in an isosceles triangle, in which the difference between the largest and smallest angles is 6°?

A8 The four square tiles having the designs as shown can be arranged to create a closed loop.
How many distinct closed loops, including the one shown here, can be made from the tiles?
(The tiles may be rotated, but a rotation of a loop is not considered distinct. A loop need not use all four tiles and may not use more than one of each type).



A9 Abbie, Betty and Clara write names on bookmarks sold for charity.
Abbie writes 7 names in 6 minutes, Betty writes 18 names in 10 minutes and Clara writes 23 names in 15 minutes.
If all of the girls work together at these rates, how long will it take them to write 540 names?

A10 In the diagram, JK and ML are parallel, $JK = KO = OJ = OM$ and $LM = LO = LK$.
Find the size of angle JMO.



Section B

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B1 In a sequence of six numbers, every term after the second term is the sum of the previous two terms. Also, the last term is four times the first term, and the sum of all six terms is 13.

What is the first term?

B2 The eight-digit number "*ppppqqqq*", where p and q are digits, is a multiple of 45.

What are the possible values of p?

B3 Jack and Jill went up a hill. They started at the same time, but Jack arrived at the top one-and-a-half hours before Jill. On the way down, Jill calculated that, if she had walked 50% faster and Jack had walked 50% slower, then they would have arrived at the top of the hill at the same time.

How long did Jill actually take to walk up to the top of the hill?

B4 The solution to each clue of this crossnumber is a two-digit number, not beginning with zero.

In how many different ways can the crossnumber be completed correctly?

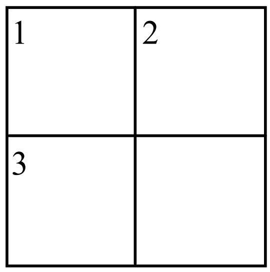
Clues

Across

1. A triangular number
3. A triangular number

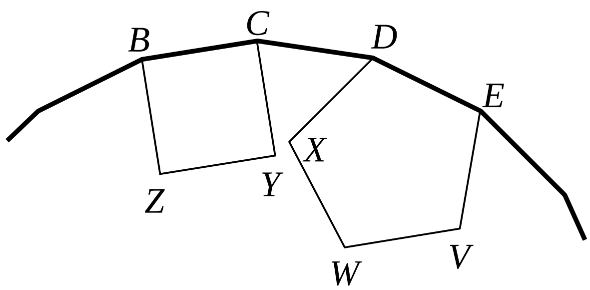
Down

1. A square number
2. A multiple of 5



B5 The diagram shows part of a regular 20-sided polygon (an icosagon) $ABCDEF\ldots$, a square $BCYZ$ and a regular pentagon $DEVWX$.

Show that the vertex X lies on the line DY.



B6 Sam has put sweets in five jars in such a way that no jar is empty and no two jars contain the same number of sweets. Also, any three jars contain more sweets in total than the total of the remaining two jars.

What is the smallest possible number of sweets altogether in the five jars?

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Tuesday 14th June 2011

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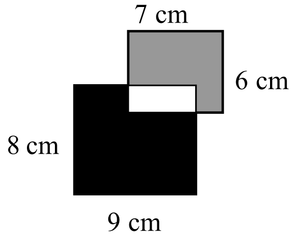
Section A

A1 What is the value of $3^3 + 3 \times 3 - 3$?

A2 Two rectangles measuring 6 cm × 7 cm and 8 cm × 9 cm overlap as shown. The region shaded grey has an area of 32 cm^2.

What is the area of the black region?



A3 Paul is 32 years old. In 10 years' time, Paul's age will be the sum of the ages of his three sons.

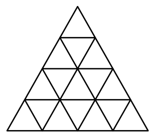
What do the ages of each of Paul's three sons add up to at present?

A4 What is the value of $\dfrac{1}{2-3} - \dfrac{4}{5-6} - \dfrac{7}{8-9}$?

A5 The base of a pyramid has n edges. In terms of n, what is the difference between the number of edges of the pyramid and the number of its faces?

A6 The diagram shows a grid of 16 identical equilateral triangles.

How many different rhombuses are there made up of two adjacent small triangles?



A7 Some rectangular sheets of paper, all the same size, are placed in a pile. The pile is then folded in half to form a booklet. The pages are then numbered in order 1, 2, 3, 4 ... from the first page to the last page.
On one of the sheets, the sum of the numbers on the four pages is 58.

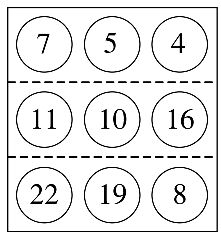
How many sheets of paper were there at the start?

A8 A puzzle starts with nine numbers placed in a grid, as shown.

On each move you are allowed to swap any two numbers.

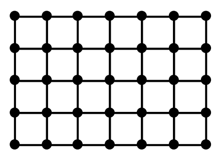
The aim is to arrange for the total of the numbers in each row to be a multiple of 3.

What is the smallest number of moves needed?



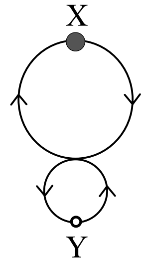
A9 The diagram represents a rectangular fishing net, made from ropes knotted together at the points shown. The net is cut several times; each cut severs precisely one section of rope between two adjacent knots.

What is the largest number of such cuts that can be made without splitting the net into two separate pieces?



A10 A 'figure of eight' track is constructed from two circles: a large circle of radius 2 units and a small circle of radius 1 unit. Two cars X, Y start out from the positions shown: X at the top of the large circular part and Y at the bottom of the small circular part. Each car travels round the complete circuit in the directions shown by the arrows.

If Y travels twice as fast as X, how far must X travel before the cars collide?



Section B

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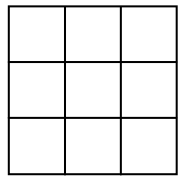
B1 Every digit of a given positive integer is either a 3 or a 4 with each occurring at least once. The integer is divisible by both 3 and 4.

What is the smallest such integer?

B2 A 3×3 grid contains nine numbers, not necessarily integers, one in each cell. Each number is doubled to obtain the number on its immediate right and trebled to obtain the number immediately below it.



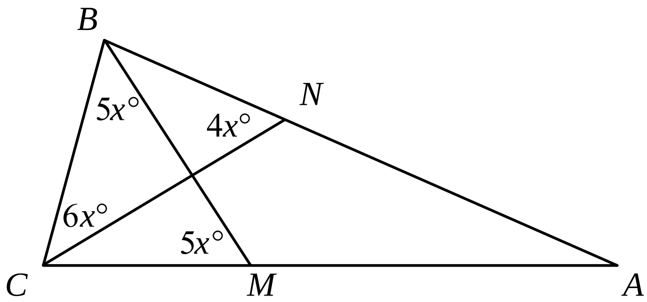
If the sum of the nine numbers is 13, what is the value of the number in the central cell?

B3 When Dad gave out the pocket money, Amy received twice as much as her first brother, three times as much as the second, four times as much as the third and five times as much as the last brother. Peter complained that he had received 30p less than Tom.

Use this information to find all the possible amounts of money that Amy could have received.

B4 In a triangle *ABC*, *M* lies on *AC* and *N* lies on *AB* so that $\angle BNC = 4x°$, $\angle BCN = 6x°$ and $\angle BMC = \angle CBM = 5x°$.

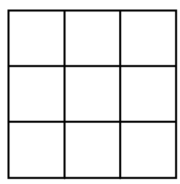
Prove that triangle *ABC* is isosceles.



B5 Calum and his friend cycle from A to C, passing through B.
During the trip he asks his friend how far they have cycled.
His friend replies "one third as far as it is from here to B".
Ten miles later Calum asks him how far they have to cycle to reach C.
His friend replies again "one third as far as it is from here to B".

How far from A will Calum have cycled when he reaches C?

B6 Pat has a number of counters to place into the cells of a 3×3 grid like the one shown. She may place any number of counters in each cell or leave some of the cells empty. She then finds the number of counters in each row and each column. Pat is trying to place counters in such a way that these six totals are all different.



What is the smallest total number of counters that Pat can use?

UK Junior Mathematical Olympiad 2012

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Tuesday 12th June 2012

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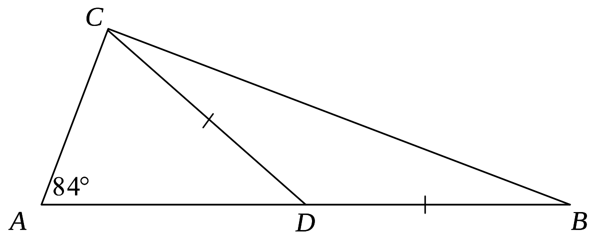
Section A

A1 What is the value of $1^1 + 2^2 + 3^3 + 4^4 - (1^4 + 2^3 + 3^2 + 4^1)$?

A2 Mike drank 60% of his glass of milk. Afterwards, 80 ml of milk remained in the glass. What volume of milk was initially in the glass?

A3 In triangle ABC, $\angle CAB = 84°$; D is a point on AB such that $\angle CDB = 3 \times \angle ACD$ and $DC = DB$.

What is the size of $\angle BCD$?

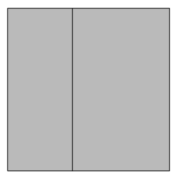


A4 A book costs £3.40 and a magazine costs £1.60. Clara spends exactly £23 on books and magazines. How many magazines does she buy?

A5 Each digit of a positive integer is 1 or 2 or 3.
Given that each of the digits 1, 2 and 3 occurs at least twice, what is the smallest such integer that is not divisible by 2 or 3?

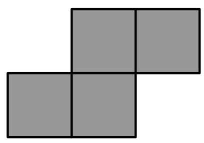
A6 A square is cut into two rectangles, as shown, so that the sum of the lengths of the perimeters of these two rectangles is 30 cm.

What is the length of a side of the square?



A7 The diagram shows a shape made from four 1×1 squares.

What is the maximum number of such shapes that can be placed inside a 5×5 square without overlapping?
(The shapes may be rotated or turned over.)



A8 An athletics club has junior (i.e. boy or girl) members and adult members. The ratio of girls to boys to adults is 3 : 4 : 9 and there are 16 more adult members than junior members. In total, how many members does the club have?

A9 What is the integer x so that $\frac{x}{9}$ lies between $\frac{71}{7}$ and $\frac{113}{11}$?

A10 A positive integer, N, has three digits and the product of its digits is also a three-digit integer. What is the smallest possible value of N?

Section B

Your solutions to Section B will have a major effect on your JMO results. Concentrate on one or two questions first and then **write out full solutions** (not just brief 'answers').

B1 There was an old woman who lived in a shoe. She had 9 children at regular intervals of 15 months. The oldest is now six times as old as the youngest. How old is the youngest child?

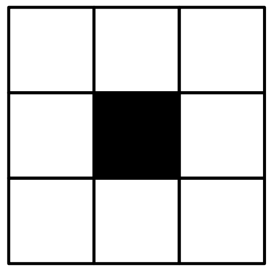
B2 Anastasia thinks of a positive integer, which Barry then doubles. Next, Charlie trebles Barry's number. Finally, Damion multiplies Charlie's number by six. Eve notices that the sum of these four numbers is a perfect square. What is the smallest number that Anastasia could have thought of?

B3 Mr Gallop has two stables which each initially housed three ponies. His prize pony, Rein Beau, is worth £250 000. Usually Rein Beau spends his day in the small stable, but when he wandered across into the large stable, Mr Gallop was surprised to find that the average value of the ponies in each stable rose by £10 000. What is the total value of all six ponies?

B4 An irregular pentagon has five different interior angles each of which measures an integer number of degrees. One angle is 76°.

The other four angles are three-digit integers which fit one digit per cell across and down into the grid on the right.

In how many different ways can the grid be completed?



B5 Three identical, non-overlapping, squares *ABCD*, *AEFG*, *AHIJ* (all labelled anticlockwise) are joined at the point *A*, and are 'equally spread' (so that $\angle JAB = \angle DAE = \angle GAH$). Calculate $\angle GBH$.

B6 The integer 23173 is such that

(a) every pair of neighbouring digits, taken in order, forms a prime number;

and (b) all of these prime numbers are different.

What is the largest integer which meets these conditions?

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Tuesday 11th June 2013

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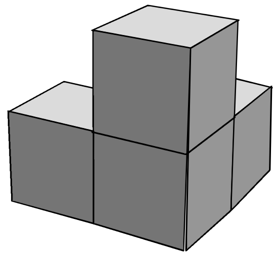
Section A

A1 What is the value of $\sqrt{3102 - 2013}$?

A2 For how many three-digit positive integers does the product of the digits equal 20?

A3 The solid shown is made by gluing together four 1 cm × 1 cm × 1 cm cubes.

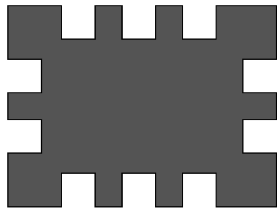
What is the total surface area of the solid?



A4 What percentage of $\frac{1}{4}$ is $\frac{1}{5}$?

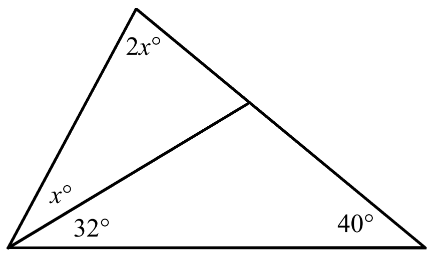
A5 Sue has a rectangular sheet of paper measuring 40 cm × 30 cm. She cuts out ten squares each measuring 5 cm × 5 cm, as shown. In each case, exactly one side of the square lies along a side of the rectangle and none of the cut-out squares overlap.

What is the perimeter of the resulting shape?



A6 I want to write a list of integers containing two square numbers, two prime numbers, and two cube numbers. What is the smallest number of integers that could be in my list?

A7 Calculate the value of x in the diagram shown.



A8 The area of a square is 0.25 m^2. What is the perimeter of the square, in metres?

A9 Each interior angle of a quadrilateral, apart from the smallest, is twice the next smaller one. What is the size of the smallest interior angle?

A10 A cube is made by gluing together a number of unit cubes face-to-face.
The number of unit cubes that are glued to exactly four other unit cubes is 96.

How many unit cubes are glued to exactly five other unit cubes?

Section B

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B1 How many numbers less than 2013 are both:

(i) the sum of two consecutive positive integers; **and**

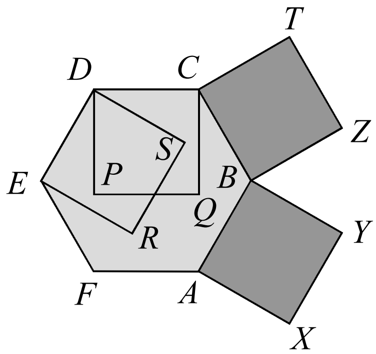
(ii) the sum of five consecutive positive integers?

B2 Pippa thinks of a number. She adds 1 to it to get a second number. She then adds 2 to the second number to get a third number, adds 3 to the third to get a fourth, and finally adds 4 to the fourth to get a fifth number.

Pippa's brother Ben also thinks of a number but he subtracts 1 to get a second. He then subtracts 2 from the second to get a third, and so on until he too has five numbers.

They discover that the sum of Pippa's five numbers is the same as the sum of Ben's five numbers. What is the difference between the two numbers of which they first thought ?

B3 Two squares $BAXY$ and $CBZT$ are drawn on the outside of a regular hexagon $ABCDEF$, and two squares $CDPQ$ and $DERS$ are drawn on the inside, as shown.



Prove that $PS = YZ$.

B4 A regular polygon P with n sides is divided into two pieces by a single straight cut. One piece is a triangle T, the other is a polygon Q with m sides.

How are m and n related?

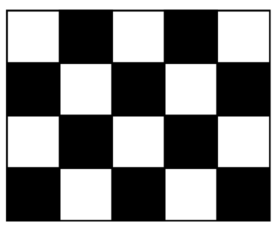
B5 Consider three-digit integers N with the two properties:

(a) N is not exactly divisible by 2, 3 or 5;

(b) no digit of N is exactly divisible by 2, 3 or 5.

How many such integers N are there?

B6 On the 4×5 grid shown, I am only allowed to move from one square to a neighbouring square by crossing an edge. So the squares I visit alternate between black and white. I have to start on a black square and visit each black square exactly once. What is the smallest number of white squares that I have to visit? Prove that your answer is indeed the smallest.



(If I visit a white square more than once, I only count it as one white square visited).

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Thursday 12th June 2014

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   Solutions must be written neatly on A4 paper. Sheets must be STAPLED together in the top left corner with the Front Sheet on top.

   Do not hand in rough work.
5. Questions A1-A10 are relatively short questions. Try to complete Section A within the first 30 minutes so as to allow well over an hour for Section B.
6. Questions B1-B6 are longer questions requiring *full written solutions*.
   This means that each answer must be accompanied by clear explanations and proofs.
   Work in rough first, then set out your final solution with clear explanations of each step.
7. These problems are meant to be challenging! Do not hurry. Try the earlier questions in each section first (they tend to be easier). Try to finish whole questions even if you are not able to do many. A good candidate will have done most of Section A and given solutions to at least two questions in Section B.
8. Answers must be FULLY SIMPLIFIED, and EXACT using symbols like π, fractions, or square roots if appropriate, but NOT decimal approximations.

DO NOT OPEN THE PAPER UNTIL INSTRUCTED BY THE INVIGILATOR TO DO SO!

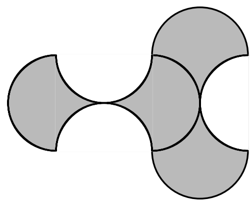
Section A

A1. What is the largest digit that appears in the answer to the calculation $(3 \times 37)^2$?

A2. What is the sum of all fractions of the form $\frac{N}{7}$, where N is a positive integer less than 7?

A3. The six angles of two different triangles are listed in decreasing order. The list starts 115°, 85°, 75° and 35°. What is the last angle in the list?

A4. The figure shows two shapes that fit together exactly. Each shape is formed by four semicircles of radius 1. What is the total shaded area?



A5. The integer 113 is prime, and its 'reverse' 311 is also prime. How many two-digit primes are there between 10 and 99 which have the same property?

A6. A square of side length 1 is drawn. A larger square is drawn around it such that all parallel sides are a distance 1 apart. This process continues until the total perimeter of the squares drawn is 144.

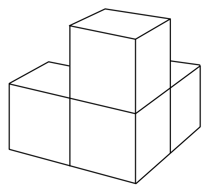
What is the area of the largest square drawn?

A7. The time is 20:14. What is the smaller angle between the hour hand and the minute hand on an accurate analogue clock?

A8. Sam has four cubes all the same size: one blue, one red, one white and one yellow. She wants to glue the four cubes together to make the solid shape shown.

How many differently-coloured shapes can Sam make?

[Two shapes are considered to be the same if one can be picked up and turned around so that it looks identical to the other.]



A9. A rectangle is made by placing together three smaller rectangles P, Q and R, without gaps or overlaps. Rectangle P measures $3 \text{ cm} \times 8 \text{ cm}$ and Q measures $2 \text{ cm} \times 5 \text{ cm}$.

How many possibilities are there for the measurements of R?

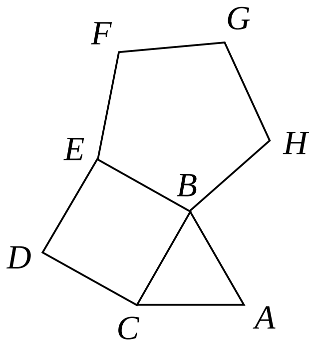
A10. My four pet monkeys and I harvested a large pile of peanuts. Monkey A woke in the night and ate half of them; then Monkey B woke and ate one third of what remained; then Monkey C woke and ate one quarter of the rest; finally Monkey D ate one fifth of the much diminished remaining pile. What fraction of the original harvest was left in the morning?

Section B

Your solutions to Section B will have a major effect on your JMO results. Concentrate on one or two questions first and then **write out full solutions** (not just brief 'answers').

B1. The figure shows an equilateral triangle *ABC*, a square *BCDE*, and a regular pentagon *BEFGH*.

What is the difference between the sizes of $\angle ADE$ and $\angle AHE$?

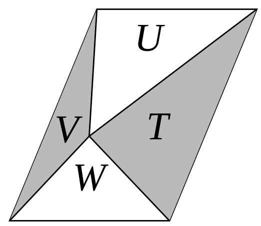


B2. I start at the square marked A and make a succession of moves to the square marked B. Each move may only be made downward or to the right. I take the sum of all the numbers in my path and add 5 for every black square I pass through.

How many paths give a sum of 51?

A	■	12	■	10
■	11	■	11	■
10	■	10	■	15
■	11	■	14	■
10	■	13	■	B

B3. A point lying somewhere inside a parallelogram is joined to the four vertices, thus creating four triangles *T*, *U*, *V* and *W*, as shown.



Prove that

$$\text{area } T + \text{area } V = \text{area } U + \text{area } W.$$

B4. There are 20 sweets on the table. Two players take turns to eat as many sweets as they choose, but they must eat at least one, and never more than half of what remains. The loser is the player who has no valid move.

Is it possible for one of the two players to force the other to lose? If so, how?

B5. Find a fraction $\frac{m}{n}$, with *m* not equal to *n*, such that all of the fractions

$$\frac{m}{n}, \frac{m+1}{n+1}, \frac{m+2}{n+2}, \frac{m+3}{n+3}, \frac{m+4}{n+4}, \frac{m+5}{n+5}$$

can be simplified by cancelling.

B6. The sum of four different prime numbers is a prime number. The sum of some pair of the numbers is a prime number, as is the sum of some triple of the numbers. What is the smallest possible sum of the four prime numbers?

UK Junior Mathematical Olympiad 2010 Solutions

A1 **55** $\frac{1}{1} + \frac{2}{\frac{1}{2}} + \frac{3}{\frac{1}{3}} + \frac{4}{\frac{1}{4}} + \frac{5}{\frac{1}{5}} = 1 + 2 \times 2 + 3 \times 3 + 4 \times 4 + 5 \times 5 = 1 + 4 + 9 + 16 + 25 = 55.$

A2 **3 : 8** Since y is common to both ratios, we change the ratios so that $x : y = 1 : 2 = 3 : 6$ and $y : z = 3 : 4 = 6 : 8$. Then we have $x : z = 3 : 8$.

A3 **14** We can note that $20^{10} = (2 \times 10)^{10} = 2^{10} \times 10^{10}$. Since $2^{10} = 1024$ has 4 digits, and multiplying by 10^{10} adds 10 zeros to the end, Tom writes down 14 digits.

A4 $\mathbf{\frac{1}{3}}$ The table shows the ways in which the monkeys (B, H and L) can select the hats. Let the hats of B, H and L be b, h and l respectively.

Monkeys			
B	H	L	
b	h	l	
b	l	h	
h	b	l	
h	l	b	*
l	h	b	
l	b	h	*

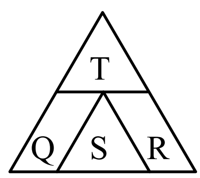
None of the monkeys have the same hat as when they arrived in only two of the six ways (*), hence the required probability is

$$\frac{2}{6} = \frac{1}{3}.$$

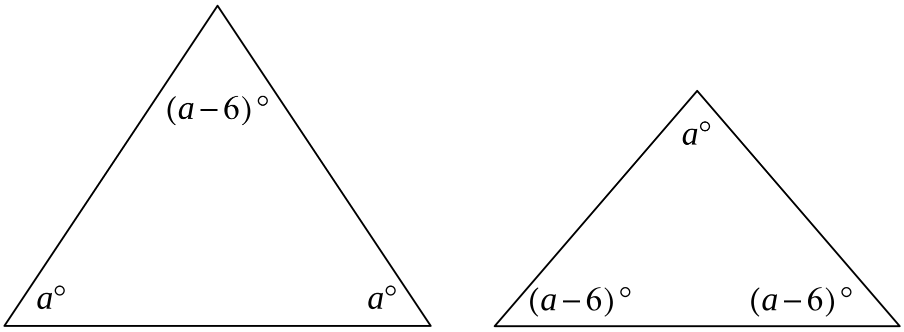
[*Alternatively*: There are $3 \times 2 \times 1 = 6$ possible ways to choose the three hats. There are two hats that B could choose. If B chose h, then L would have to choose b and H would have to choose l. If B chose l, then H would have to choose b and L would have to choose h. So once B has chosen his hat the other two are fixed. So there are just the two possible alternatives out of the six ways. So the probability is $\frac{2}{6} = \frac{1}{3}$.]

A5 **2010** Let the two numbers be a and b, where $a > b$. Then we have $a + b = 97$ and $a - b = 37$. Hence $2a = 134$ and therefore $a = 67$ and $b = 30$. The product of 67 and 30 is 2010.

A6 **1 : 1** Let us call the large triangle P. Since triangles T and S are congruent, they have the same height, which is half the height of P. Thus the area of each of T and S is a quarter of the area of P. Therefore parallelograms Q and R together form the other half and thus each occupies a quarter of P. So R and T are equal in area.

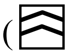


A7 64° Let the largest angle be $a°$, whence the smallest angle is $(a - 6)°$. There are two possibilities, shown in the diagrams below.



In the first we have $3a - 6 = 180$, so $a = 62$.
In the second we have $3a - 12 = 180$, so $a = 64$.
Thus the largest possible angle in such a triangle is 64°.

A8 13 To create a closed loop, one must use  at one end and  at the other.

Let us assume that the loop starts with  (turned this way) and ends with the , in some orientation (,  or ).
There is just 1 loop that uses only these tiles. If one tile is put between them, there are two ways in which each of the other two tiles can connect them ( or  and  or ). So there are 4 loops with three tiles.

Using all four tiles, there are two orders in which  and  can be placed, and, there are two possible orientations for each of these tiles, making $2 \times 2 \times 2 = 8$ ways in all. Hence there are $1 + 4 + 8 = 13$ possible loops altogether.

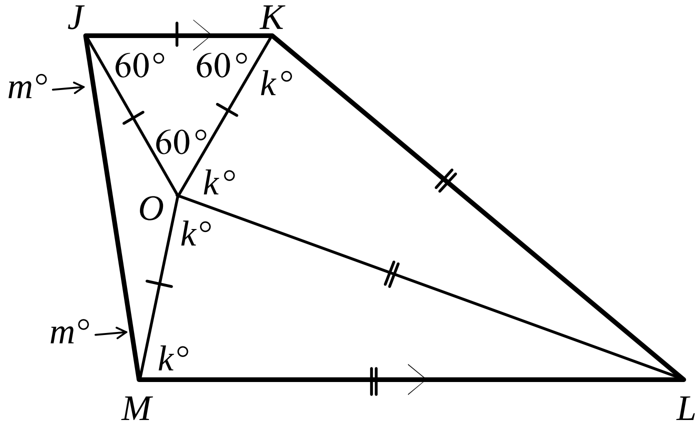
A9 2 hours Since the lowest common multiple of 6, 10 and 15 is 30, we can say that in 30 minutes Abbie writes $7 \times \frac{30}{6} = 35$ cards, Betty writes $18 \times \frac{30}{10} = 54$ cards, and Clara writes $23 \times \frac{30}{15} = 46$ cards. So together they write $35 + 54 + 46 = 135$ cards in half an hour. Thus the time taken to write 540 cards is $\frac{540}{135}$ = 4 half-hours = 2 hours.

A10 20° Since triangle JKO is equilateral, $\angle JOK = \angle KJO = \angle JKO = 60°$.

Let $\angle JMO = m°$. Then, since JMO is an isosceles triangle, $\angle MJO = m°$ and $\angle JOM = (180 - 2m)°$.

Let $\angle OKL = k°$ and so, since KLO is an isosceles triangle, $\angle LOK = k°$.

Triangles KLO and OLM are congruent (SSS), and so $\angle MOL = \angle OML = k°$.



Now taking angles at point O, we have $180 - 2m + 60 + 2k = 360$, whence $k = m + 60$.

Since JK is parallel to ML, $\angle KJM + \angle JML = 180°$ and so $(60 + m) + (m + k) = 180$.

Hence $180 = 60 + 2m + k = 60 + 2m + m + 60 = 3m + 120$, so $m = 20$, i.e. $\angle JMO = 20°$.

B1 In a sequence of six numbers, every term after the second term is the sum of the previous two terms. Also, the last term is four times the first term, and the sum of all six terms is 13.

What is the first term?

Solution

Let the first and second terms be a and b respectively. Then we derive the sequence

$$a,\ b,\ a + b,\ a + 2b,\ 2a + 3b,\ 3a + 5b.$$

We know that the last term is four times the first term, so $3a + 5b = 4a$. Therefore $a = 5b$ and so the sequence is

$$5b,\ b,\ 6b,\ 7b,\ 13b,\ 20b.$$

The sum of these is 13, so $52b = 13$, $b = \frac{13}{52} = \frac{1}{4}$ and $a = 5 \times \frac{1}{4} = \frac{5}{4}$. Thus the first term is $1\frac{1}{4}$.

B2 The eight-digit number "*ppppqqqq*", where p and q are digits, is a multiple of 45.

What are the possible values of p?

Solution

It might be argued that there is a trivial solution where $p = q = 0$. It is, however, usual to assume that numbers do not begin with zeros and so we shall proceed assuming that $p \neq 0$.

We first observe that every multiple of 45 is a multiple of both 5 and 9, and also that p and q are single-digit integers. Applying the usual rules of divisibility by 5 and 9 to the number *ppppqqqq* we deduce that $q = 0$ or $q = 5$ and that $4p + 4q$ is a multiple of 9.
In the case $q = 0$, $4p$ is a multiple of 9, hence $p = 9$.
In the case $q = 5$, $4p + 20 = 4(p + 5)$ is a multiple of 9. Therefore $p + 5$ is a multiple of 9. Hence $p = 4$.
(Thus there are two possible numbers: 99 990 000 and 44 445 555.)

B3 Jack and Jill went up a hill. They started at the same time, but Jack arrived at the top one-and-a-half hours before Jill. On the way down, Jill calculated that, if she had walked 50% faster and Jack had walked 50% slower, then they would have arrived at the top of the hill at the same time.

How long did Jill actually take to walk up to the top of the hill?

Solution

Let t be the number of hours that Jill took to the top of the hill.
So the time taken by Jack was $\left(t - 1\frac{1}{2}\right)$ hours.
If Jack had walked 50% more slowly, he would have taken twice as long, ie. $(2t - 3)$ hours.
If Jill had walked 50% faster, she would have taken $\frac{2}{3}$ of the time, ie. $\frac{2}{3}t$ hours.
So we know that $\frac{2}{3}t = 2t - 3$, whence $2t = 6t - 9$ and so $t = \frac{9}{4} = 2\frac{1}{4}$.
Hence Jill took $2\frac{1}{4}$ hours.

B4 The solution to each clue of this crossnumber is a two-digit number, not beginning with zero.

In how many different ways can the crossnumber be completed correctly?

Clues

Across

1. A triangular number
3. A triangular number

Down

1. A square number
2. A multiple of 5

1	2
3	

Solution

We start by listing the two-digit triangular numbers and two-digit square numbers:

triangular numbers:	10, 15, 21, 28, 36, 45, 55, 66, 78, 91
square numbers:	16, 25, 36, 49, 64, 81.

Since 2 Down is a multiple of 5, it ends in either 0 or 5.

Hence 3 Across ends in either 0 or 5 and there are four such triangular numbers: 10, 15, 45, and 55. In each case there is only one possible square number at 1 Down, as shown in the following figures:

8	
1	0

(a)

8	
1	5

(b)

6	
4	5

(c)

2	
5	5

(d)

Now consider 1 Across, a triangular number. In (a) and (b), there is no two-digit triangular number whose first digit is 8, and hence we can rule out cases (a) and (b).

In (c), the only triangular number whose first digit is 6 is 66. In (d), there are two triangular numbers whose first digit is 2, namely 21 and 28.

Therefore there are three different ways in which the crossnumber can be completed:

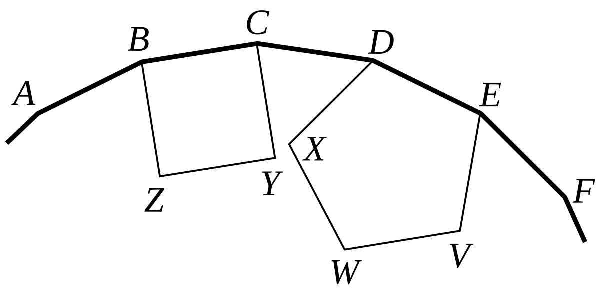
6	6
4	5

2	1
5	5

2	8
5	5

B5 The diagram shows part of a regular 20-sided polygon (an icosagon) $ABCDEF\ldots$, a square $BCYZ$ and a regular pentagon $DEVWX$.

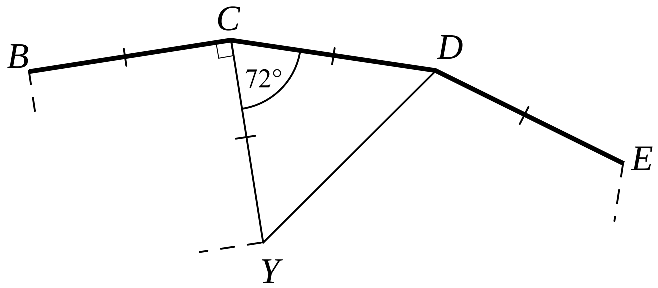
Show that the vertex X lies on the line DY.



Solution

Considering the interior angles of the square, the regular pentagon, and the regular icosagon, $\angle BCY = 90°$, $\angle EDX = \left(180 - \frac{360}{5}\right)° = 108°$ and $\angle BCD = \left(180 - \frac{360}{20}\right)° = 162°$.

Now $\angle DCY = (162 - 90)° = 72°$ and also $\angle CDX = (162 - 108)° = 54°$.
Now consider triangle CDY.



Since the icosagon is regular, $BC = CD$ and, as $BCYZ$ is a square, $BC = CY$.

Therefore $CD = CY$ and CDY is an isosceles triangle.

Hence $\angle CDY = \frac{1}{2}(180 - 72)° = 54°$.

However, as observed above, $\angle CDX = 54°$ and so $\angle CDX = \angle CDY$.

Thus we can conclude that point X lies on the line DY.

B6 Sam has put sweets in five jars in such a way that no jar is empty and no two jars contain the same number of sweets. Also, any three jars contain more sweets in total than the total of the remaining two jars.

What is the smallest possible number of sweets altogether in the five jars?

Solution

Let the number of sweets in the five jars be a, b, c, d and e, where $a < b < c < d < e$. Since $d > c > b$, and b, c and d are integers, $d \geqslant b + 2$. Similarly $e \geqslant c + 2$.

Now, since any three jars contain more sweets in total than the total of the remaining two jars, in particular $a + b + c > d + e$, and so $a + b + c > b + 2 + c + 2$, hence $a > 4$.

Try $a = 5$. The smallest possible values of the other numbers are 6, 7, 8 and 9, which give a total of 35. Because 5, 6 and 7, the three smallest numbers, give a total of 18, which is over half of 35, any other selection of three of these numbers will have a total greater than that of the remaining two numbers.

Thus the smallest total is 35.

UK Junior Mathematical Olympiad 2011 Solutions

A1 33 $3^3 + 3 \times 3 = 27 + 9 - 3 = 33$.

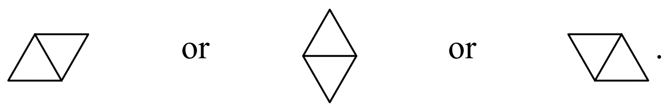
A2 62 cm² Since the area of the 6 cm × 7 cm rectangle is $42\ \text{cm}^2$, the area of the white rectangle is $(42 - 32)\ \text{cm}^2 = 10\text{cm}^2$. Hence the area of the black region is $(8 \times 9 - 10)\ \text{cm}^2 = 62\text{cm}^2$.

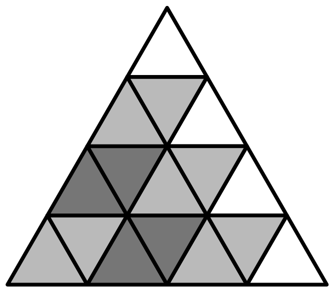
A3 12 In 10 years' time, Paul will be 42 and the sum of the ages of his three sons will be 3×10 years $= 30$ years more than it is now. So the sum of the ages of each of his three sons now is $(42 - 30)$ years $= 12$ years.

A4 10 $\frac{1}{2-3} - \frac{4}{5-6} - \frac{7}{8-9} = \frac{1}{-1} - \frac{4}{-1} - \frac{7}{-1} = (-1) - (-4) - (-7) = -1 + 4 + 7 = 10$.

A5 $n - 1$ A pyramid whose base has n edges also has n edges rising to its apex and hence $2n$ edges in total. It also has $n + 1$ faces, including the base. So the difference between the number of edges of the pyramid and the number of its faces is $2n - (n + 1) = n - 1$.

A6 18 A rhombus formed from a pair of adjacent triangles is in one of three orientations:





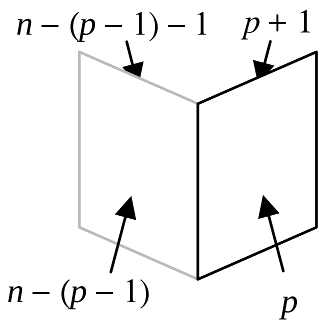
It can be seen that there are 6 rhombi in the first orientation. By symmetry, there are 6 in each of the other two, giving a total of 18.

[*Alternatively*, each rhombus is determined by its short diagonal; the short diagonals are the sides of the interior triangles, so there are $3 \times 6 = 18$.]

A7 7 Let the number of pages in the booklet be n.
Then the page numbers on the outside sheet are 1, 2, $n - 1$ and n and have a total of

$$1 + 2 + (n - 1) + n = 2n + 2.$$

In general, on a sheet whose lowest numbered page is page p, the next page is $p + 1$. Because page p has $(p - 1)$ pages before it, the highest numbered page (on the same side as page p) has $(p - 1)$ pages after it, and so is page $n - (p - 1)$. The remaining page is likewise page $n - (p - 1) - 1$.

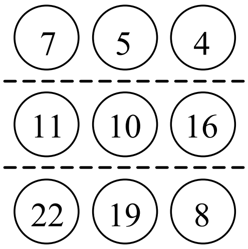


The total of the four page numbers of a sheet is therefore

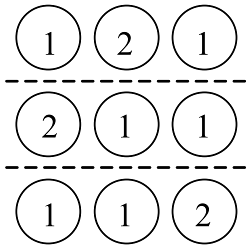
$$p + (p + 1) + [n - (p - 1)] + [n - (p - 1) - 1] = 2n + 2.$$

Hence $2 + 2n = 58$ and so $n = 28$. Since each piece of paper used provides four pages of the booklet, the number of sheets used is seven.

A8 2 Since we are interested only in whether or not each of the rows can have a total which is a multiple of 3, we can reduce each of the numbers to their remainder on division by three. So

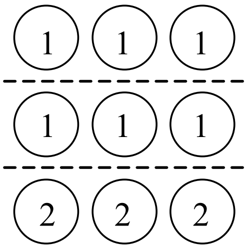


becomes

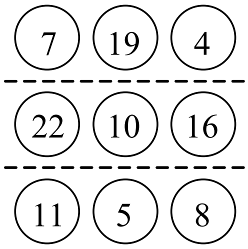


.

Since the row totals are 4, 4 and 4, it is now clear that making one swap will not achieve three totals each of which is a multiple of 3. However, if we swap both the 2s in the first two rows with the 1s in the last row, we get



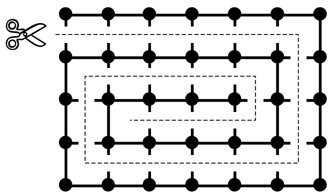
which corresponds to



.

A9 24 In order to stay connected, a network like this will have a minimum required number of sections of rope, since if we start with just one knot, each additional knot requires at least one additional section of rope. So the smallest possible number of sections of rope to connect n knots is $n - 1$.

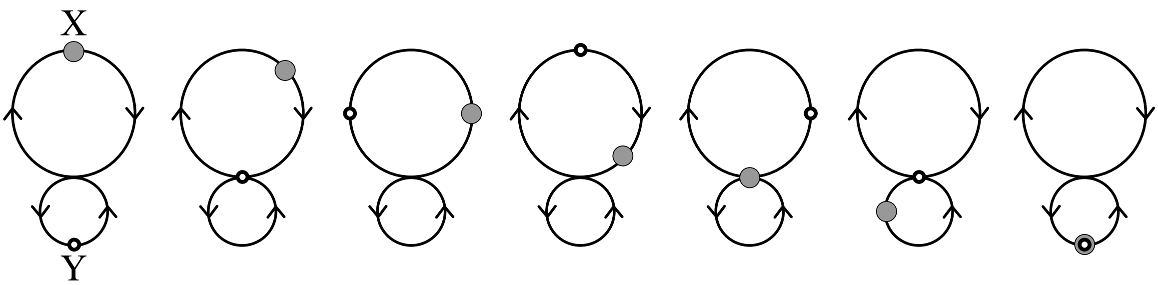
For this net there are 35 knots, and we will need at least 34 sections of rope left to keep them all connected and the diagram on the right shows one way in which we can leave just 34 sections of rope.



It can be seen that the net starts with 5×6 horizontal sections and 7×4 vertical sections, giving a total of $30 + 28 = 58$.

Hence we can make a maximum of $58 - 34 = 24$ cuts.

A10 3π The only way for X and Y to collide is by Y catching up X or by both being at the crossing point at the same moment. Since the speed of car X is half that of car Y, car X will travel half way around the circuit in the time car Y takes to complete a full circuit, as the diagrams below illustrate, going from left (the start) to right (the first collision).



The length of the circumference of the larger circle is 4π units, and that of the smaller circle is half as long, 2π units.

Therefore X will have travelled $\frac{1}{2} \times 4\pi + \frac{1}{2} \times 2\pi = 3\pi$ units.

B1 Every digit of a given positive integer is either a 3 or a 4 with each occurring at least once. The integer is divisible by both 3 and 4.

What is the smallest such integer?

Solution

Let the integer be n. It is evident that n must have more than two digits, since none of 3, 4, 34 or 43 are divisible by by both 3 and 4.

In order for n to be divisible by 3, the sum of its digits must be a multiple of 3.

In order for n to be divisible by 4, its last two digits have to be a multiple of 4. Given that all the digits can be only 3 or 4, it is clear that the last two digits of n can only be 44.

If n were a three-digit number, the hundreds digit would have to be 3 (since the other digits are both 4); however, 344 is not a multiple of 3.

If n were a four-digit number, the smallest value we can consider is 3344, but this is not a multiple of 3. The next smallest is 3444 and this satisfies all the criteria of the problem.

B2 A 3×3 grid contains nine numbers, not necessarily integers, one in each cell. Each number is doubled to obtain the number on its immediate right and trebled to obtain the number immediately below it.

If the sum of the nine numbers is 13, what is the value of the number in the central cell?

Solution

If we let the number in the top left-hand corner cell be a, we can write the other numbers in terms of a, as shown in the diagram:

a	$2a$	$4a$
$3a$	$6a$	$12a$
$9a$	$18a$	$36a$

The sum of these nine numbers is $91a$.

Given that the sum is 13, we have $91a = 13$ and so $a = 13 \div 91 = \frac{1}{7}$.

Thus the number in the central cell is $6a = 6 \times \frac{1}{7} = \frac{6}{7}$.

B3 When Dad gave out the pocket money, Amy received twice as much as her first brother, three times as much as the second, four times as much as the third and five times as much as the last brother. Peter complained that he had received 30p less than Tom.

Use this information to find all the possible amounts of money that Amy could have received.

Solution

As we are considering halves, thirds, quarters and fifths, we shall let Amy receive $60x$ pence.

Then	her first brother receives	$30x$ pence,
	her second brother receives	$20x$ pence,
	her third brother receives	$15x$ pence
and	her last brother receives	$12x$ pence.

We do not know which of the brothers are Peter and Tom, though Peter is younger than Tom. We can now tabulate the six possibilities:

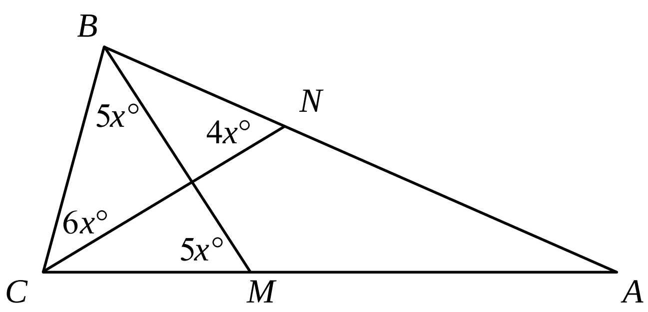
	brothers				comparison	value
case	first	second	third	fourth		of x
A	$30x$	$20x$			$30x - 20x = 30$	$x = 3$
B	$30x$		$15x$		$30x - 15x = 30$	$x = 2$
C	$30x$			$12x$	$30x - 12x = 30$	$x = 1\frac{2}{3}$
D		$20x$	$15x$		$20x - 15x = 30$	$x = 6$
E		$20x$		$12x$	$20x - 12x = 30$	$x = 3\frac{3}{4}$
F			$15x$	$12x$	$15x - 12x = 30$	$x = 10$

We can eliminate two of these cases: in case C, the value of $x = 1\frac{2}{3}$ would mean that the second brother received $20 \times 1\frac{2}{3} = 33\frac{1}{3}$ pence; in case E, the value of $x = 3\frac{3}{4}$ would mean that the third brother received $15 \times 3\frac{3}{4} = 56\frac{1}{4}$ pence.

The remaining four cases mean that Amy could have received £1.20, £1.80, £3.60 or £6.00.

B4 In a triangle ABC, M lies on AC and N lies on AB so that $\angle BNC = 4x°$, $\angle BCN = 6x°$ and $\angle BMC = \angle CBM = 5x°$.

Prove that triangle ABC is isosceles.



Solution

By considering the sum of the angles of triangle NBC, we find that

$$\angle NBC = (180 - 6x - 4x)° = (180 - 10x)°.$$

Similarly, by considering the sum of the angles of triangle MCB, we find that

$$\angle MCB = (180 - 5x - 5x)° = (180 - 10x)°.$$

Hence $\angle ABC = \angle NBC = (180 - 10x)° = \angle MCB = \angle ACB$, and so triangle ABC is isosceles.

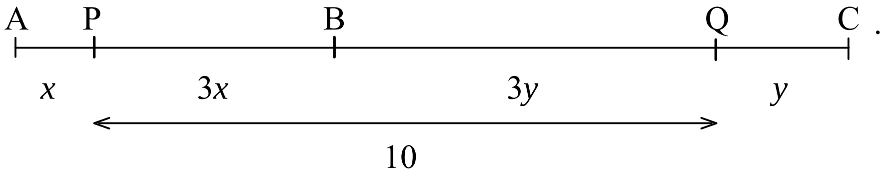
B5 Calum and his friend cycle from A to C, passing through B.
During the trip he asks his friend how far they have cycled.
His friend replies "one third as far as it is from here to B".
Ten miles later Calum asks him how far they have to cycle to reach C.
His friend replies again "one third as far as it is from here to B".

How far from A will Calum have cycled when he reaches C?

Solution

Let the points at which Calum asked each question be P and Q.
Let the distances AP and QC be x miles and y miles respectively.
Then the distance from P to B is $3x$ miles and the point P must lie between A and B.
Similarly the distance from B to Q is $3y$ miles.

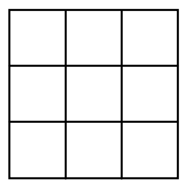
Thus



Since they have cycled 10 miles between P and Q, we know that $3x + 3y = 10$, and so

$$AC = 4x + 4y = \tfrac{4}{3}(3x + 3y) = \tfrac{4}{3} \times 10 = 13\tfrac{1}{3} \text{ miles}.$$

B6 Pat has a number of counters to place into the cells of a 3×3 grid like the one shown. She may place any number of counters in each cell or leave some of the cells empty. She then finds the number of counters in each row and each column. Pat is trying to place counters in such a way that these six totals are all different.



What is the smallest total number of counters that Pat can use?

Solution

Let n be the smallest number of counters that Pat can use.

We observe first that n = the sum of the three row totals
= the sum of the three column totals
whence the sum of all six totals = $2n$.

The smallest possible different totals are 0, 1, 2, 3, 4 and 5, so that the least that the sum of the six totals could be is $0 + 1 + 2 + 3 + 4 + 5 = 15$.

Hence $2n \geqslant 15$ and so $n \geqslant 8$.

Consider $n = 8$. It is indeed possible to find a way for Pat to place the counters in such a way that all six totals are all different – two possible configurations are shown, with the totals, on the right:

			0
	1	1	2
1	2	3	6
1	3	4	

			0
	2		2
1	1	4	6
1	3	4	

Therefore the smallest number of counters that Pat can use is 8.

UK Junior Mathematical Olympiad 2012 Solutions

A1 266 $1 + 4 + 27 + 256 - (1 + 8 + 9 + 4) = 288 - 22 = 266$.

A2 200ml The remaining 40% has volume 80ml. So 10% of the volume is 20 ml and 100% of the volume is 200ml.

A3 27° Let $\angle ACD$ be $x°$. Then $\angle CDB$ is $3x°$.

Then, from the straight line ADB, $\angle ADC$ is $(180 - 3x)°$.

Consider the triangle ADC with angle sum 180°, $84 + x + (180 - 3x) = 180$, so $x = 42$.

Hence $\angle BCD = \angle DBC = \frac{1}{2}(180 - 3x)°$ which is 27°.

Alternative: Label $\angle ACD$ as $x°$, which gives $\angle CDB$ as $3x°$. Then using the property that an exterior angle is the sum of the two opposite interior angles, we have the equation $3x = x + 84$, giving $x = 42$.

Hence, as above, $\angle BCD = \angle DBC = \frac{1}{2}(180 - 3x)°$ which is 27°.

A4 8 Let the number of books bought be b and the number of magazines bought be m. Then (working in pence): $2300 = 340b + 160m$. This simplifies to $115 = 17b + 8m$ (*). But $17 \times 7 = 119$ which is greater than 115 so we know $b < 7$ and we can also see from (*) that b must be an odd number. Hence $b = 1, 3$ or 5.

If $b = 1$ then $115 = 17 + 8m$ so $m = 98/8$ which is not a whole number.

If $b = 3$ then $115 = 51 + 8m$ so $m = 64/8 = 8$.

If $b = 5$ then $115 = 85 + 8m$ so $m = 30/8$ which is not a whole number.

Therefore the only possible value of b is 3 which gives $m = 8$.

A5 1112233 An integer is divisible by 3 when the sum of its digits is divisible by 3.

Since $2 \times (1 + 2 + 3) = 12$ is divisible by 3, the digits 1, 1, 2, 2, 3, 3 are not sufficient. For the smallest possible integer, we choose an extra '1'.

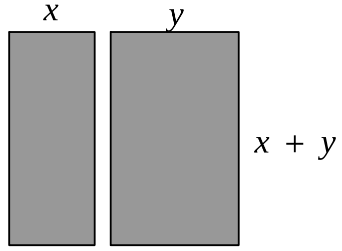
The small digits must be at the front to have the smallest integer overall. So the smallest integer made from these digits is 1112233.

A6 5cm Let the short sides of the rectangles be x cm and y cm.

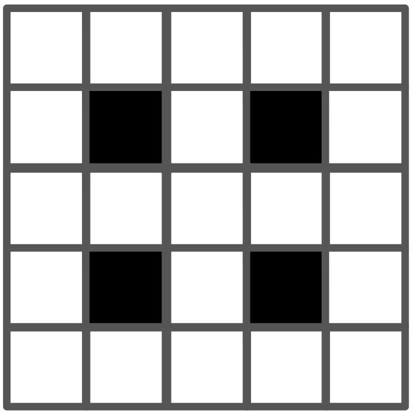
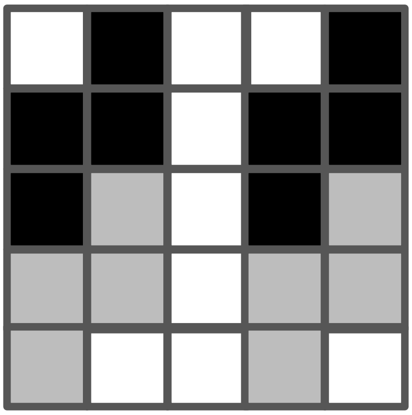
So the side of the square is $(x + y)$ cm.

Then the total perimeter is $6(x + y)$ cm.

The side of the square is therefore $30/6 = 5$ cm.



A7 **4**

Colour four cells of the 5 by 5 grid black, as shown in the first diagram.

Then any shape placed on the grid must cover a black cell, no matter how the shape is placed. But there are only four black cells, so the maximum number of shapes that may be placed is four. There are many ways in which this maximum can be achieved, such as the one shown on the right.

A8 **128** Nine-sixteenths of the total club members are adults and seven-sixteenths are junior. So two-sixteenths of the total, the difference between the number of adults and juniors, is sixteen. Thus one-sixteenth of the total membership is 8.

The total membership is therefore $16 \times 8 = 128$.

A9 **92** We require $\frac{x}{9} > \frac{71}{7}$, that is $7x > 639$, that is $x > 91\frac{2}{7}$.

We require $\frac{x}{9} < \frac{113}{11}$, that is $11x < 1017$, that is $x < 92\frac{5}{11}$.

Since x is an integer, $x = 92$.

A10 **269** Suppose N is 'abc'.

Then a is at least 2, otherwise the product is at most $1 \times 9 \times 9 = 81$, and so does not have three digits.

Consider numbers of the form '2bc'. Then b is at least 6, otherwise the product is at most $2 \times 5 \times 9 = 90$, and so does not have three digits.

Consider numbers of the form '26c'. Then c is at least 9, otherwise the product is at most $2 \times 6 \times 8 = 96$, and so does not have three digits.

Now the product of the digits of 269 is $2 \times 6 \times 9 = 108$, so 269 is the smallest value of N.

B1 There was an old woman who lived in a shoe. She had 9 children at regular intervals of 15 months. The oldest is now six times as old as the youngest. How old is the youngest child?

Solution

Let the youngest child be aged x years. Then the oldest child's age is $x + 8 \times \frac{15}{12} = x + 10$. So $x + 10 = 6x$. Then $5x = 10$. Thus $x = 2$, so the youngest child is 2.

B2 Anastasia thinks of a positive integer, which Barry then doubles. Next, Charlie trebles Barry's number. Finally, Damion multiplies Charlie's number by six. Eve notices that the sum of these four numbers is a perfect square. What is the smallest number that Anastasia could have thought of?

Solution

Let Anastasia's integer be a. Then the four numbers are a, $2a$, $6a$ and $36a$. The sum of these numbers is $45a$. Now $45 = 3 \times 3 \times 5 = 9 \times 5$. So $45a$ being a square means that $5a$ must be a square because 9 is already a square. So to be the smallest, $5a$ must be 25. Thus the smallest a is 5.

B3 Mr Gallop has two stables which each initially housed three ponies. His prize pony, Rein Beau, is worth £250 000. Usually Rein Beau spends his day in the small stable, but when he wandered across into the large stable, Mr Gallop was surprised to find that the average value of the ponies in each stable rose by £10 000. What is the total value of all six ponies?

Solution

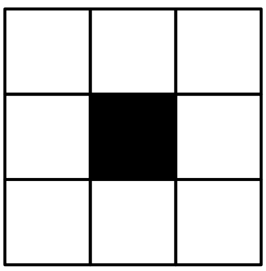
At the start, let the value of the three ponies in the small stable be £$(s + 250\,000)$ and the value of the other three ponies in the large stable be £ l.

Then $\frac{s}{2} - \frac{s + 250\,000}{3} = 10\,000$ and so $s = 560\,000$. Also $\frac{l + 250\,000}{4} - \frac{l}{3} = 10\,000$ and so $l = 630\,000$.

Therefore the total value of all six ponies is £$(s + l + 250\,000) = £1\,440\,000$.

B4 An irregular pentagon has five different interior angles each of which measures an integer number of degrees. One angle is 76°.

The other four angles are three-digit integers which fit one digit per cell across and down into the grid on the right.



In how many different ways can the grid be completed?

Solution

The sum of the interior angles of a pentagon is 540°.

The remaining four angles therefore add to 464°.

The integer 1 must appear in both top corners and in the left-hand bottom corner. We see that we need to find integers a, b, c, d and y all less than or equal to 9 so that '1a1' + '1by' + '1cy' + '1d1' = 464. So $1 + y + y + 1 = 4$ or 14 since y is at most 9.

1	a	1
d		b
1	c	y

Consider $2 + 2y = 4$, then $y = 1$ and we now need to find different single digit integers a, b, c and d so that $a + b + c + d = 6$. The only possible values of a, b, c, d are 0, 1, 2, 3. There are four ways of choosing a, then three ways of choosing b, then two ways of choosing c and only one way then left for d. In this case, there are $4 \times 3 \times 2 \times 1 = 24$ ways of completing the grid.

Consider $2 + 2y = 14$, then $y = 6$ and we now need to find $a + b + c + d = 5$ with $a \neq d$ and $b \neq c$. The only possible sets of integers are 0, 0, 1, 4; 0, 0, 2, 3; 0, 1, 1, 3 and 0, 1, 2, 2. Each of these sets can be used in 8 ways. For example,

a	0	0	0	0	1	1	4	4
b	0	0	1	4	0	4	0	1
c	1	4	0	0	4	0	1	0
d	4	1	4	1	0	0	0	0

making a total of another 32 ways.

Altogether, there are 56 possible ways of completing the grid.

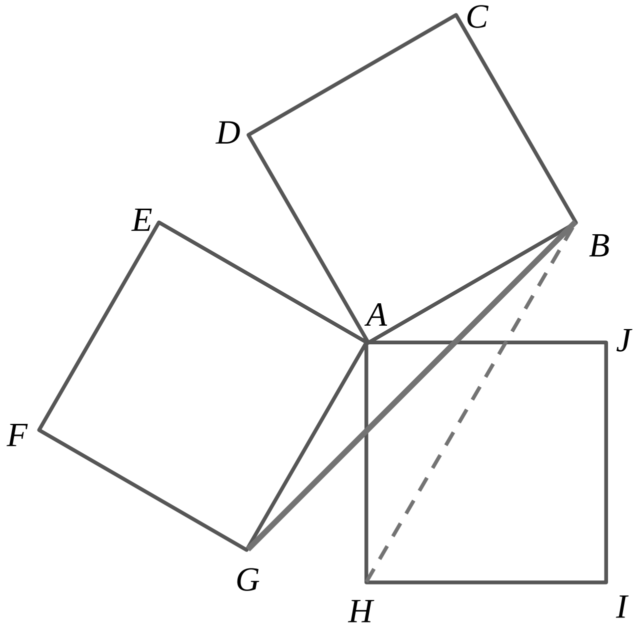
B5 Three identical, non-overlapping squares *ABCD*, *AEFG*, *AHIJ* (all labelled anticlockwise) are joined at the point *A*, and are 'equally spread' (so that $\angle JAB = \angle DAE = \angle GAH$). Calculate $\angle GBH$.

Solution

Since the squares are 'equally spread' $\angle JAB = \angle DAE = \angle GAH = 30°$. Hence $\angle GAB = \angle GAH + \angle HAJ + \angle JAB = 30° + 90° + 30° = 150°$. Triangle *GAB* is isosceles since $GA = GB$.

Therefore $\angle GBA = \angle BGA = 15°$.

Similarly $\angle HAB = \angle HAJ + \angle JAB = 90° + 30° = 120°$. Triangle *BAH* is isosceles since $BA = HA$. Therefore $\angle AHB = \angle ABH = 30°$ which gives $\angle GBH = \angle ABH - \angle GBA = 30° - 15° = 15°$.



Observe that the diagram is not the only valid arrangement so the answer of 15° is not unique.

B6 The integer 23173 is such that

(a) every pair of neighbouring digits, taken in order, forms a prime number;

and (b) all of these prime numbers are different.

What is the largest integer which meets these conditions?

Solution

No two-digit prime ends in 5 or an even digit, so that 2, 4, 6, 8 or 5 can only appear as the first digit of the required number.

There are ten two-digit prime numbers with two odd digits that do not include 5. We may list them in a table, putting those with the same first digit in the same row, and those with the same second digit in the same column:

11	13	17	19
31		37	
71	73		79
		97	

Now any 12-digit number which contains all of these ten primes and has a digit 2, 4, 6, 8, or 5 at the front will clearly be larger than any failing to do so. And no number of the required form can have more than 12 digits. Let us assume that we can find a number of this form, that is, containing all ten primes in the table and with first digit 2, 4, 6, 8, or 5, and try to construct the largest such number.

Notice that any digit x not at an end of the number corresponds to two primes ax and xb, so that the digit x is both the first digit of a prime and the second digit of a prime.

From the table, we see that the digit 1 occurs four times as a first digit, and only three times as a second digit, so that if we are to use all these primes, one of those of the form '1_' does not appear as part of a pair from the table. The only possible way this can happen is for the number to start 'd1_', where d is 5 or even.

Similarly, for all the primes containing a digit 9 to appear, the number has to end '_9'.

Now the largest prime of the form 'd1', where d is 5 or even, is 61. And the largest number of the form '61e' with '1e' prime is 619. This leaves 79 as the only possible last pair of digits. So we now have a number of the form 619 _ _ _ _ _ _ _ _ 79. From the table the only option now is to use 97 at the start, giving a number of the form 619 7 _ _ _ _ _ _79.

To have the largest possible answer, there is only one choice for the digit before the 7 at the end, so we have
619 7 _ _ _ _ _ 179.

We may continue to add the largest possible digit to the end of those at the start. The sequence continues:

619	73 _	_ _ _	179
619	737	_ _ _	179
619	737	1 _ _	179
619	737	13 _	179
619	737	131	179

The final number uses all ten primes from the table, together with the largest possible first digit of the numbers that do so. When constructing the number, we have also used the largest possible next digit at each stage.
Therefore 619 737 131 179 is the largest number of the required form.

UK Junior Mathematical Olympiad 2013 Solutions

A1 33 $3102 - 2013 = 1089 = 9 \times 121 = 3^2 \times 11^2 = 33^2$. Therefore $\sqrt{3102 - 2013} = \sqrt{33^2} = 33$.

A2 9 First, we need to find triples of digits whose product is $20 = 2^2 \times 5$.
The only possible triples are $\{1, 4, 5\}$ and $\{2, 2, 5\}$.
There are 6 possible ways of ordering the digits: $\{1, 4, 5\}$.
There are 3 possible ways of ordering the digits: $\{2, 2, 5\}$.
Therefore the total number of 3-digit numbers for which the product of the digits is equal to 20 is 9.

A3 18 cm² The surface area of four such cubes arranged separately is $4 \times 6\ \text{cm}^2 = 24\ \text{cm}^2$. However, in this solid, there are three pairs of faces that overlap and so do not contribute to the surface area of the solid.
Therefore, the total surface area is $(24 - 3 \times 2 \times 1)\ \text{cm}^2 = 18\ \text{cm}^2$.

Alternatively, a bird's eye view from each of six directions has surface area $3\ \text{cm}^2$. So the total surface area is $6 \times 3 = 18\ \text{cm}^2$.

A4 80% Note that $\frac{1}{5} : \frac{1}{4} = \frac{4}{20} : \frac{5}{20} = 4 : 5 = 80 : 100$. Hence $\frac{1}{5}$ is 80% of $\frac{1}{4}$.
Alternatively $\frac{1}{5} \div \frac{1}{4} = \frac{4}{5}$; and $\frac{4}{5} \times 100 = 80$. Therefore $\frac{1}{5}$ is 80% of $\frac{1}{4}$.

A5 240 cm The perimeter of the original paper is $40 + 40 + 30 + 30 = 140$ cm.
Each cut-out square adds 10 cm to the perimeter.
So the final perimeter is $140 + 10 \times 10 = 240$ cm.

A6 4 A prime number cannot be a square or cube.
Hence there must be at least 4 numbers in the list.
We can find a list with two sixth powers (i.e. both squares and cubes) and two prime numbers e.g. $1^6, 2, 3, 2^6$ or $2^6, 3^6, 5, 7$ (where 2, 3, 5, 7 are all prime).
So the smallest number of integers in my list is 4.

A7 36 Since the angles in a triangle add up to 180°, we have $2x + (x + 32) + 40 = 180$.
This simplifies to $3x + 72 = 180$, which has the solution $x = 36$.

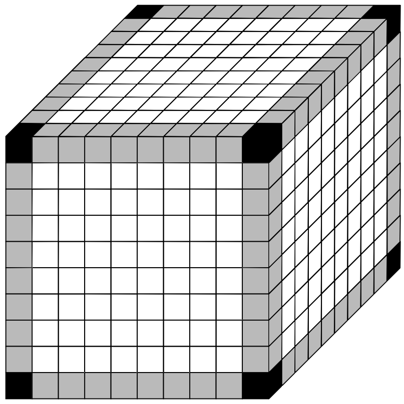
A8 2 m The length of each side equals 0.5 m since $0.5^2 = 0.25$.
Hence the perimeter is $4 \times 0.5 = 2$ m.

A9 **24°** Let the angles of the quadrilateral be $x°$, $2x°$, $4x°$ and $8x°$.

The sum of angles in a quadrilateral is 360°.

Thus $15x = 360$ which gives $x = 24$.

A10 **384** Let the side length of the large cube be n.



On each edge of the large cube, there are $n - 2$ cubes glued to exactly 4 other cubes, shown shaded grey.

So in total there are $12(n - 2)$ cubes glued to exactly 4 other cubes.

Therefore $12(n - 2) = 96$ which gives $n = 10$.

On each face of the large cube, there are 8^2 cubes glued to exactly 5 other cubes, which are unshaded.

So in total, there are $6 \times 64 = 384$ cubes glued to exactly five other unit cubes.

B1 How many numbers less than 2013 are both:

(i) the sum of two consecutive positive integers; **and**

(ii) the sum of five consecutive positive integers?

Solution

A number satisfies condition (i) if and only if it is of the form

$$n + (n + 1) = 2n + 1$$

for $n \geqslant 1$, i.e. it is an odd number greater than or equal to 3.

A number satisfies condition (ii) if and only if it is of the form

$$(m - 2) + (m - 1) + m + (m + 1) + (m + 2) = 5m$$

for some $m \geqslant 3$, i.e. it is a multiple of 5 greater than or equal to 15.

So a number satisfies both conditions if and only if it is of the form $5p$ with p an odd number and $p \geqslant 3$; i.e. $p = 2q + 1$ for $q \geqslant 1$.

Now $5(2q + 1) \leqslant 2013$ implies that $q \leqslant 200$. So there are 200 such numbers. satisfying both conditions.

B2 Pippa thinks of a number. She adds 1 to it to get a second number. She then adds 2 to the second number to get a third number, adds 3 to the third to get a fourth, and finally adds 4 to the fourth to get a fifth number.

Pippa's brother Ben also thinks of a number but he subtracts 1 to get a second. He then subtracts 2 from the second to get a third, and so on until he too has five numbers.

They discover that the sum of Pippa's five numbers is the same as the sum of Ben's five numbers. What is the difference between the two numbers of which they first thought ?

Solution

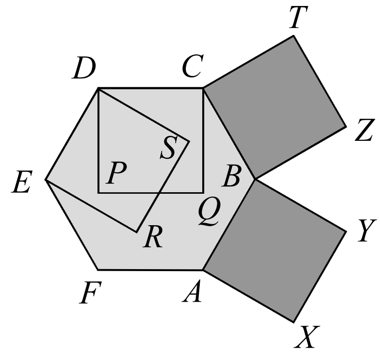
Let Pippa's original number be p and Ben's be b.

Then $p + (p + 1) + (p + 1 + 2) + (p + 1 + 2 + 3) + (p + 1 + 2 + 3 + 4)$
$= b + (b - 1) + (b - 1 - 2) + (b - 1 - 2 - 3) + (b - 1 - 2 - 3 - 4)$.

This simplifies first to $5p + 20 = 5b - 20$ and then to $8 = b - p$.

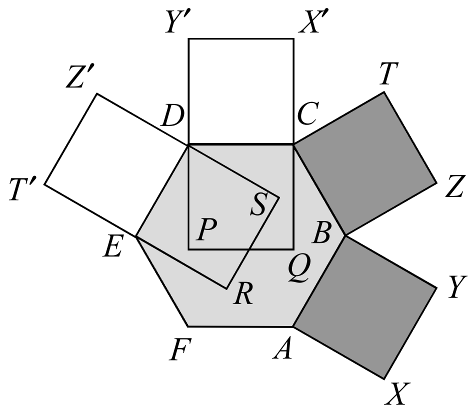
Hence the difference between the original numbers is 8.

B3 Two squares $BAXY$ and $CBZT$ are drawn on the outside of a regular hexagon $ABCDEF$, and two squares $CDPQ$ and $DERS$ are drawn on the inside, as shown.



Prove that $PS = YZ$.

Solution 1



Draw squares $EDZ'T'$ and $DCX'Y'$ on the outside of the hexagon.

Since $ABCDEF$ is regular, angles EDC and ABC are both 120° and also angles EDZ', CDY', CBZ and ABY are all right angles so $\angle Y'DZ' = \angle YBZ = 60°$. Also the lengths of the sides of the four squares are all equal as they are equal to the sides of the regular hexagon. Thus triangles $DY'Z'$ and BYZ are congruent (SAS) and hence $Z'Y' = ZY$.

Now compare triangles $Y'DZ'$ and PDS.

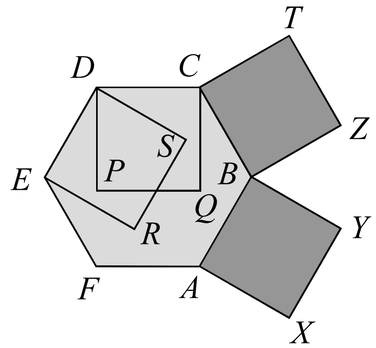
$Y'D = PD$ and $Z'D = DS$. (These are the same length as the sides of the hexagon.)

Angle $Y'DZ'$ = Angle PDS as they are vertically opposite angles.

So triangle $Y'DZ'$ is congruent to triangle PDS.

So $PS = Z'Y' = ZY$.

Solution 2



In a regular hexagon, an interior angle is 120°. In a square an interior angle is 90°.

Consider ΔBZY. $BZ = CB$ as both are sides of the square $BZTC$. $CB = AB$ as both are sides of the regular hexagon. $AB = BY$ as both are sides of the square $BAXY$. Therefore $BZ = BY$.

Since angles at a point total 360°, it follows that $90° + 120° + 90° + \angle ZBY = 360°$ and so $\angle ZBY = 60°$.

Therefore ΔBZY is an isosceles triangle with an angle of 60° between the equal sides and so is an equilateral triangle.

In a similar way, consider ΔDPS. $DP = DC$ as both are sides of the square $DPQC$; $DC = DE$ as both are sides of the regular hexagon and $DE = DS$ as both are sides of the square $DERS$. Hence $DP = DS$.

$\angle EDC = 120°$ and $\angle PDC = 90°$ hence $\angle EDP = 30°$. $\angle EDC = 120°$ and $\angle EDS = 90°$ hence $\angle SDC = 30°$. $\angle EDC = \angle EDP = \angle PDS = \angle SDC$ so $\angle PDS = 60°$.

Therefore ΔPDS is an isosceles triangle with an angle of 60° between the equal sides and so is an equilateral triangle.

Also $DP = DC = CB = BZ$ so the sides of the two equilateral triangles are the same length.

Therefore ΔDPS and ΔBYZ are exactly the same size (*called congruent triangles*).

Therefore $PS = YZ$.

B4 A regular polygon P with n sides is divided into two pieces by a single straight cut. One piece is a triangle T, the other is a polygon Q with m sides.

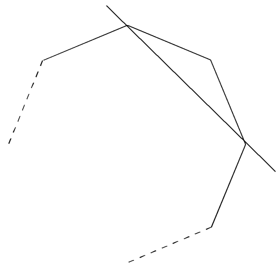
How are m and n related?

Solution

There are three possible ways in which one straight cut can create a triangle.

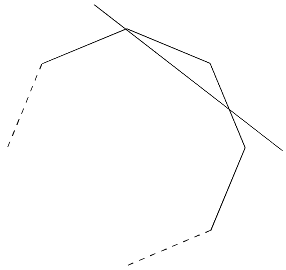
Case 1: The straight cut goes through two vertices of the polygon.

Then $m = n - 1$.



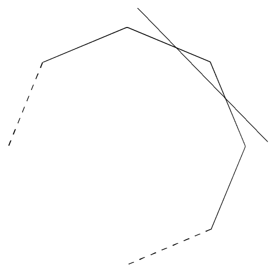
Case 2: The straight cut goes through exactly one vertex of the polygon.

Then $m = n$.



Case 3: The straight cut goes through no vertices of the polygon.

Then $m = n + 1$.



B5 Consider three-digit integers N with the two properties:

(a) N is not exactly divisible by 2, 3 or 5;

(b) no digit of N is exactly divisible by 2, 3 or 5.

How many such integers N are there?

Solution

Condition (b) means that each digit of N must be either 1 or 7 (since 0, 2, 4, 6, 8 are divisible by 2; 0, 5 are divisible by 5 and 3, 6, 9 are divisible by 3).

A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

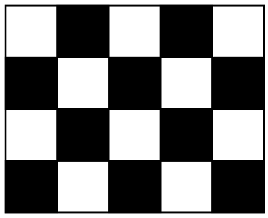
But

$$1 + 1 + 1 = 3 \qquad 1 + 1 + 7 = 9 \qquad 1 + 7 + 7 = 15 \qquad 7 + 7 + 7 = 21$$

which are all divisible by 3.

Hence there are no 3-digit numbers N satisfying both these conditions.

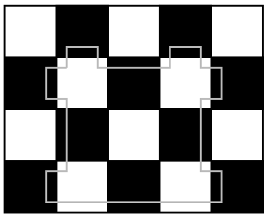
B6 On the 4 × 5 grid shown, I am only allowed to move from one square to a neighbouring square by crossing an edge. So the squares I visit alternate between black and white. I have to start on a black square and visit each black square exactly once. What is the smallest number of white squares that I have to visit? Prove that your answer is indeed the smallest.



(If I visit a white square more than once, I only count it as one white square visited).

Solution

It is possible to visit each black square exactly once by travelling through 4 white squares (as shown in this diagram).



Suppose there is a route using only three white squares. The maximum number of black squares adjacent to the first white square on the route is 4. To reach the second white square on the route, the route must pass via one of those black squares − and so there are no more than 3 additional black squares adjacent to the second white square. Likewise, when the third white square is reached there are at most 3 additional black squares adjacent to it. This means that with 3 white squares we can reach at most 10 black squares and, moreover, we can only reach 10 if each of the three white squares is adjacent to 4 black squares. However, in the given diagram, there are only three such white squares − and none of them is adjacent to the black squares in the bottom corners. Hence three white squares is not enough. This means that the smallest number of white squares I have to visit is four.

UK Junior Mathematical Olympiad 2014 Solutions

A1 **3** Firstly, $3 \times 37 = 111$ and so $(3 \times 37)^2 = 111^2$. Now

$$\begin{aligned} 111^2 &= 1 \times 111 + 10 \times 111 + 100 \times 111 \\ &= 111 + 1110 + 11100 \\ &= 12\,321. \end{aligned}$$

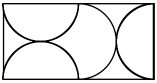
Therefore the largest digit is 3.

A2 **3** The sum in question is

$$\frac{1}{7} + \frac{2}{7} + \frac{3}{7} + \frac{4}{7} + \frac{5}{7} + \frac{6}{7} = \frac{21}{7} = 3.$$

A3 **20°** First note that $115° + 85° > 180°$ and $115° + 75° > 180°$ so one triangle contains both the 75° and the 85° angles. Also note that $85° + 75° + 35° > 180°$ so that triangle does not contain the 35° angle. Hence one triangle must have internal angles including 85° and 75°, and the other triangle must have internal angles 115° and 35°. The two remaining angles are therefore $180° - (115° + 35°) = 30°$ and $180° - (85° + 75°) = 20°$. Therefore the last angle in the list is 20°.

A4 **8** The shapes can be cut and rearranged to make a 4×2 rectangle as shown.



Therefore the shaded area is 8.

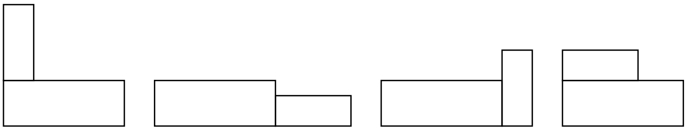
A5 **9** Any number ending in 2, 4, 6 or 8 is even. Similarly, any number ending in 5 is divisible by 5. Hence, for both a two-digit number and its reverse to be a prime, the original number can only start with 1, 3, 7 or 9. There are 10 two-digit primes starting with 1, 3, 7 or 9, namely 11, 13, 17, 19, 31, 37, 71, 73, 79 and 97 and, of these, only 19 does not have its reverse in the list. Hence there are 9 two-digit primes with the desired property.

A6 **121** The squares have side lengths 1, 3, 5, 7, 9, 11, . . . and so the sums of the perimeters are 4, 16, 36, 64, 100, 144, ... Thus the largest square has side-length 11 and area 121.

A7 **163°** The minute hand takes 60 minutes to make a complete turn, and so rotates through $360° \div 60 = 6°$ in one minute. Therefore, at 14 minutes past the hour, the minute hand has rotated by $14 \times 6° = 84°$. The hour hand takes 12 hours, or 720 minutes, to make a complete turn and so rotates through 0.5° in one minute. Therefore, at 20:14, the hour hand has rotated through $240° + 7° = 247°$. Thus the angle between the minute hand and the hour hand is $247° - 84° = 163°$.

A8 **8** The 'corner' cube may be chosen in four ways. Given a choice of the 'corner' cube, there are then three choices for the top cube and a further two choices for the left-hand cube. This gives $4 \times 3 \times 2 = 24$ different ways of arranging the cubes. However, the shape can be rotated so that each of the three faces of the 'corner' cube that are not joined to any other cube are at the bottom and the shape would then look the same. So the set of 24 arrangements contains groups of three that can be rotated into each other. Hence the number of differently coloured shapes is $24 \div 3 = 8$.

A9 **4** The rectangles *P* and *Q* must be placed together edge-to-edge in one of the following ways.

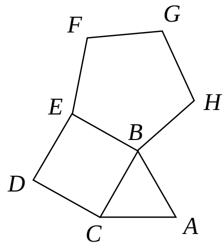


Therefore there are 4 possibilities for the measurements of *R*: 6×5, 1×5, 8×2 and 3×2.

A10 $\mathbf{\frac{1}{5}}$ After Monkey A has eaten half of the pile, the fraction of the original pile that remains is $\frac{1}{2}$. Monkey B eats $\frac{1}{3}$ of the remaining pile and so leaves $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$ of the original pile. Monkey C leaves $\frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$; and Monkey D leaves $\frac{4}{5} \times \frac{1}{4} = \frac{1}{5}$ of the original pile.

B1 The figure shows an equilateral triangle *ABC*, a square *BCDE*, and a regular pentagon *BEFGH*.

What is the difference between the sizes of $\angle ADE$ and $\angle AHE$?



Solution

We calculate the sizes of $\angle ADE$ and $\angle AHE$ in turn. Since *ABC* is an equilateral triangle, $\angle ACB = 60°$. Since *BCDE* is a square, $\angle BCD = 90°$. As edge *BC* is shared by the triangle and the square, $AC = CD$. Therefore the triangle *ACD* is isosceles. Now $\angle ACD = 60° + 90° = 150°$ and so $\angle ADC = 15°$. Therefore

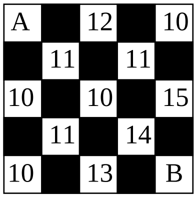
$\angle ADE = \angle EDC - \angle ADC = 90° - 15° = 75°$.

Now, for angle $\angle AHE$, $\angle EBH = 108°$ as *BEFGH* is a regular pentagon. By considering the angles around *B*, $\angle ABH = 360° - (108° + 90° + 60°) = 102°$. Since triangle *ABH* is isosceles, this means that $\angle AHB = 39°$. Also, triangle *HBE* is isosceles and so $\angle BHE = 36°$. Therefore $\angle AHE = \angle AHB + \angle BHE = 39° + 36° = 75°$.

So the difference between the sizes of the angles is zero.

B2 I start at the square marked A and make a succession of moves to the square marked B. Each move may only be made downward or to the right. I take the sum of all the numbers in my path and add 5 for every black square I pass through.

How many paths give a sum of 51?

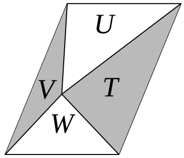


Solution

Any path from A to B must pass through four black squares, contributing 20 to the sum.

To have a path with sum 51, the numbers in the remaining three squares must sum to 31. Since all the numbers in the squares have two digits, the only possible way to make a sum of 31 is $10 + 10 + 11$. However any path must pass through the diagonal containing the numbers 13, 14 and 15. Thus there are no paths giving a sum of 51.

B3 A point lying somewhere inside a parallelogram is joined to the four vertices, thus creating four triangles T, U, V and W, as shown.

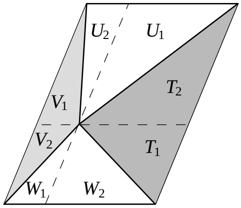


Prove that

$$\text{area } T + \text{area } V = \text{area } U + \text{area } W.$$

Solution

The parallelogram may also be split into four parallelograms, each having the point as a vertex.



If we label the separate triangles formed as shown in the diagram then it can be seen that area V_1 = area U_2, area U_1 = area T_2, area T_1 = area W_2 and area W_1 = area V_2.

Therefore

$$\begin{aligned}\text{area } T + \text{area } V &= \text{area } T_1 + \text{area } T_2 + \text{area } V_1 + \text{area } V_2\\ &= \text{area } W_2 + \text{area } U_1 + \text{area } U_2 + \text{area } W_1\\ &= \text{area } U_1 + \text{area } U_2 + \text{area } W_1 + \text{area } W_2\\ &= \text{area } U + \text{area } W.\end{aligned}$$

B4 There are 20 sweets on the table. Two players take turns to eat as many sweets as they choose, but they must eat at least one, and never more than half of what remains. The loser is the player who has no valid move.

Is it possible for one of the two players to force the other to lose? If so, how?

Solution

The losing player is the one who is left with 1 sweet on the table, because taking that sweet would mean taking more than half of what remains. The first player can force the second to lose by leaving 15, 7, 3 and 1 sweets after successive turns. Call the first player A and the second player B. On her first turn, A should leave 15 sweets. Then B must leave between 8 and 14 sweets (inclusive). No matter how many sweets are left, A should leave 7 on her next turn. This will always be possible as 7 is at least half of the number of sweets remaining. Next, player B must leave between 4 and 6 sweets. Player A can then leave 3 sweets as 3 is at least half of the number of sweets remaining. Player B must now take 1 sweet, leaving 2 on the table. Finally, player A takes 1 sweet leaving player B with no valid move.

B5 Find a fraction $\frac{m}{n}$, with m not equal to n, such that all of the fractions

$$\frac{m}{n}, \frac{m+1}{n+1}, \frac{m+2}{n+2}, \frac{m+3}{n+3}, \frac{m+4}{n+4}, \frac{m+5}{n+5}$$

can be simplified by cancelling.

Solution

Suppose that $n > m$ and write $n = m + k$, where k is an integer. Then the six fractions are

$$\frac{m}{m+k}, \frac{m+1}{(m+1)+k}, \frac{m+2}{(m+2)+k}, \frac{m+3}{(m+3)+k}, \frac{m+4}{(m+4)+k}, \frac{m+5}{(m+5)+k}.$$

These fractions can all be cancelled provided that k is a multiple of each of the integers

$$m, m+1, m+2, m+3, m+4, m+5.$$

For example, take $m = 2$. Then k must be a common multiple of 2, 3, 4, 5, 6, 7; say $k = 420$. Then the six fractions are $\frac{2}{422}, \frac{3}{423}, \frac{4}{424}, \frac{5}{425}, \frac{6}{426}, \frac{7}{427}$; so $m = 2$ and $n = 422$ is a solution.

B6 The sum of four different prime numbers is a prime number. The sum of some pair of the numbers is a prime number, as is the sum of some triple of the numbers. What is the smallest possible sum of the four prime numbers?

Solution

One of the four primes must be 2. This is because the sum of four odd positive integers is even and bigger than 2, so cannot be prime. Similarly, 2 must be used in the pair. But 2 must not be used in the triple, otherwise its sum would be even and greater than 2.

The triple must sum to a prime that is also 2 smaller than a prime, so that the four chosen numbers sum to a prime. The sum of the three smallest odd primes is $3 + 5 + 7 = 15$, which is not prime, and so the sum of the triple must be greater than 15. The possible sums are therefore 17, 29, 41, ... In order to have sum 17, one of the numbers 3, 5 or 7 must be increased by 2. However, 3 and 5 cannot be increased by 2 as this would mean the primes in the triple are not all different, and 7 cannot be increased by 2 as 9 is not prime. Thus the triple cannot have sum 17. It is possible, however, to find three primes that sum to 29. For example, 5, 7 and 17.

Therefore the smallest possible sum of the four primes is $29 + 2 = 31$. (And an example of four primes with all of the desired properties is $\{2, 5, 7, 17\}$; the pair could then be $\{2, 5\}$and the triple $\{5, 7, 17\}$.)