Junior Mathematical Olympiad

Past Papers and Solutions 2005-2009

United Kingdom Mathematics Trust

UK Junior Mathematical Olympiad 2005

Organised by The United Kingdom Mathematics Trust

Tuesday 14th June 2005

RULES AND GUIDELINES : READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING

1. Time allowed: 2 hours.
2. **The use of calculators *and measuring instruments* is forbidden.**
3. All candidates must be in *School Year 8 or below* (England and Wales), *S2 or below* (Scotland), *School Year 9 or below* (Northern Ireland).
4. For questions in Section A *only the answer is required*. Enter each answer neatly in the relevant box on the Front Sheet. Do not hand in rough work.

   For questions in Section B you must give *full written solutions*, including clear mathematical explanations as to why your method is correct.

   Solutions must be written neatly on A4 paper. Sheets must be STAPLED together in the top left corner with the Front Sheet on top.

   Do not hand in rough work.
5. Questions A1-A10 are relatively short questions. Try to complete Section A within the first 45 minutes so as to allow well over an hour for Section B.
6. Questions B1-B6 are longer questions requiring *full written solutions*.
   This means that each answer must be accompanied by clear explanations and proofs.
   Work in rough first, then set out your final solution with clear explanations of each step.
7. These problems are meant to be challenging! Do not hurry. Try the earlier questions in each section first (they tend to be easier). Try to finish whole questions even if you can't do many. A good candidate will have done most of Section A and given solutions to at least two questions in Section B.
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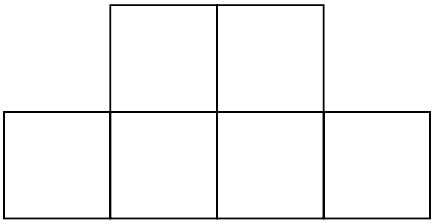
Section A

A1 How many seconds are there in one fortieth of an hour?

A2 The diagram shows a shape made from six squares, each of side 1cm.

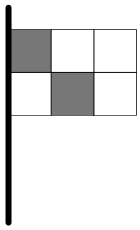
Four copies of the shape are placed together (without leaving any holes or having any overlaps) to form a rectangle.

What is the perimeter of the rectangle?



A3 Three different integers have a sum of 1 and a product of 36. What are they?

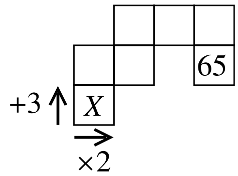
A4 A picture of a flag is to be completed by shading two squares which do not share an edge. The diagram shows one way in which this can be done.
How many different possible completed pictures are there (including the one shown)?



A5 In this puzzle, when you move up one square you **add 3**, when you move down one square you **subtract 3** and when you move to the right one square you **multiply by 2**.

The last square contains the number 65.

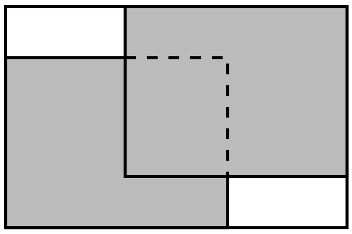
What number is in the square marked X ?



A6 Charlie's factory makes crème eggs and caramel eggs. The crème eggs are produced by a machine at the rate of 30 per minute, while the caramel eggs are produced by a different machine at the rate of 40 per minute. On a day when these two machines were in operation for a combined time of 18 hours, 36 000 eggs were produced in total. For how many hours was the crème egg machine in use?

A7 A sheet of paper is exactly the same size as a rectangular table top. The paper is cut in half and the two halves are placed on the table as shown.

What is the ratio of the area of table left uncovered (white) to the area which is covered twice?



A8 A large container holds 14 litres of a solution which is 25% antifreeze, the remainder being water. How many litres of antifreeze must be added to the container to make a solution which is 30% antifreeze?

A9 Colin has a collection of more than 24 coins. When he puts the coins in piles of 6, there are 3 coins remaining. When he puts the coins in piles of 8, there are 7 coins remaining. How many coins remain when he puts the coins in piles of 24?

A10 A closed rectangular box is a 'double cube', in which the top and bottom are squares, and the height is twice the width. The greatest distance between any two points of this box is 9 cm. What is the total surface area of the box?

Section B

Your solutions to Section B will have a major effect on the JMO results. Concentrate on one or two questions first and then **write out full solutions** (not just brief 'answers').

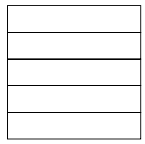
B1 The first three terms of a sequence are $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$. The fourth term is $\frac{1}{2} - \frac{1}{3} + \frac{1}{4}$; henceforth, each new term is calculated by taking the previous term, subtracting the term before that, and then adding the term before that.

(i) Write down the first six terms of the sequence, giving your answers as simplified fractions.

(ii) Find the 10th term and the 100th term, and explain why they have to be what you claim.

B2 The diagram shows a square which has been divided into five congruent rectangles. The perimeter of each rectangle is 51 cm.

What is the perimeter of the square?



B3

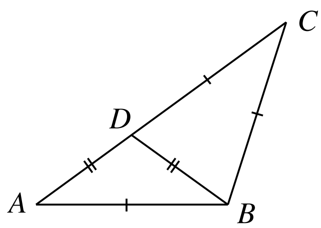
	175										70	

The diagram above is to be completed so that each box contains a whole number, the total of the numbers in the thirteen boxes is 2005 and the sum of the numbers in any three consecutive boxes is always the same.

In how many different ways is it possible to complete the diagram in this way?

B4 In this figure *ADC* is a straight line and $AB = BC = CD$. Also, $DA = DB$.

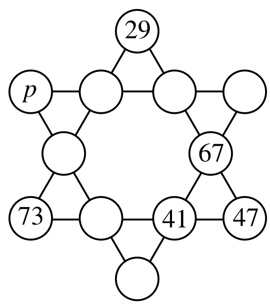
Find the size of $\angle BAC$.



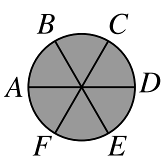
B5 In a magic hexagram, the numbers in every line of four circles have the same total. The diagram shows a magic hexagram which uses twelve different prime numbers.

Five numbers, including the smallest and the largest of the twelve primes, are shown.

Find the value of *p*, explaining the steps in your reasoning.



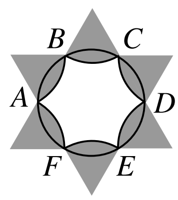
B6



Points *A*, *B*, *C*, *D*, *E* and *F* are equally spaced around a circle of radius 1. The circle is divided into six sectors as shown on the left.

The six sectors are then rearranged so that *A*, *B*, *C*, *D*, *E* and *F* lie on a new circle, also of radius 1, as shown on the right with the sectors pointing outwards.

Find the area of the curvy *unshaded* region.



UK Junior Mathematical Olympiad 2006

Organised by The United Kingdom Mathematics Trust

Tuesday 13th June 2006

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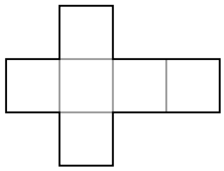
Section A

A1 What is the value of $1 + 2 \times (3 + 4^5) + 6 + 7 - 8 \times 9 + 10$?

A2 The perimeter of this net of a cube is 42cm.
What is the volume of the cube?

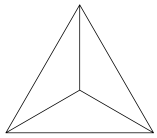


A3 Sarah writes down all the three-digit positive integers for which the product of their digits equals 36. What is the difference between the greatest and the smallest of these numbers?

A4 An equilateral triangle is drawn on a sheet of white card and divided into three identical regions as shown. Then each region is painted red or yellow or blue. More than one region may be painted in the same colour.

How many different painted triangles can be made in this way?

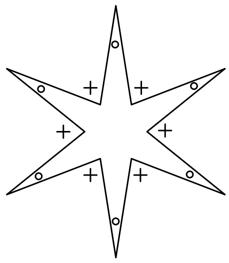
(Rotating a triangle does not make it different.)



A5 A balloon seller starts the day with a certain number of balloons. He then sells one third of his balloons to boys, 20% to girls, and three times the difference between these amounts to adults. At the end of the day, he has eight balloons left. How many balloons did the seller have at the start?

A6 In the diagram the star is made up of equal line segments.

Each of the angles marked + is 70°.
What is the size of the angles marked ∘ ?



A7 Each year on Tom's birthday, his grandfather gives him some pocket money. The amount, in pence, is calculated by multiplying together Tom's age and his grandfather's age on that day.

This year Tom received £7.81. How much did he receive last year?

A8 The numbers 1 to 9 are to be placed so that there is one number in each square and the row and column totals are as shown

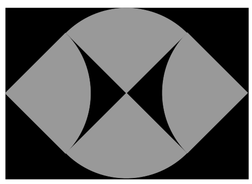
			8
			13
			24
11	14	20	

What number goes in the central square?

A9 The prime number 11 may be written as the sum of three prime numbers in two different ways: 2 + 2 + 7 and 3 + 3 + 5. What is the smallest prime number which can be written in two different ways as the sum of three prime numbers **which are all different**?

A10 In the flag shown alongside, the regions shaded grey are quarter circles.

If the height of the flag is 1m, what is its breadth?



Section B

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B1 In her purse, Jenny has 20 coins, with a total value of £5. There are three denominations of coin – 10p, 20p and 50p – in her purse and she has more 50p coins than 10p coins. How many of each type of coin does she have?

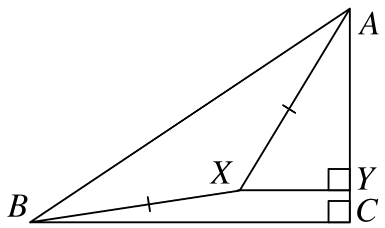
B2

$$97 \rightarrow 63 \rightarrow 18 \rightarrow 8.$$

An example of a particular type of number chain is shown above. The first number must be a positive integer. Each number after the first is the product of the digits of the previous number, so in this case $63 = 9 \times 7$; $18 = 6 \times 3$; $8 = 1 \times 8$. The chain stops when a single-digit number is reached.

Suppose that in such a chain the final number is 6. Find all possible two-digit first numbers for this chain.

B3 In this diagram, Y lies on the line AC, triangles ABC and AXY are right angled and in triangle ABX, $AX = BX$. The line segment AX bisects angle BAC and angle AXY is seven times the size of angle XBC. What is the size of angle ABC?



B4 Start with an equilateral triangle ABC of side 2 units, and construct three outward-pointing squares $ABPQ$, $BCTU$, $CARS$ on the three sides AB, BC, CA. What is the area of the hexagon $PQRSTU$?

B5 An intelligent bug starts at the point (4, 0) and follows these instructions:

(i) first face "East" and walk one unit to the point (5,0);

(ii) from then on, whenever you arrive at a point (x, y) with x and y both integers,
either turn left through 90° if $x - y$ is a multiple of 4 or is 1 more than a multiple of 4;
or turn right through 90° if $x - y$ is 2 more than a multiple of 4 or is 3 more than a multiple of 4;
and then walk one unit to the next point whose coordinates are both integers.

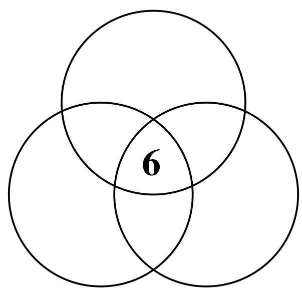
After one move, the bug is at the point (5,0).

(a) Where will the bug be after 12 moves?

(b) Where will the bug be after 50 moves?

B6 The numbers 1 to 7 are to be placed in the seven regions formed by three overlapping circles, with 6 in the central region, so that there is one number inside each region and the total of the numbers inside each circle is T.

What values of T are possible?



UK Junior Mathematical Olympiad 2007

Organised by The United Kingdom Mathematics Trust

Tuesday 12th June 2007

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Section A

A1 What is the value of $1^5 - 2^4 + 3^3 - 4^2 + 5^1$?

A2 What is the value of k if "$7k$ minutes past nine" is the same time as "$8k$ minutes to ten"?

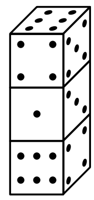
A3 Charlie boils seven eggs for his breakfast. He puts the eggs into the pan one at a time, but waits one minute after putting one egg in before putting the next egg in. If he boils each egg for three minutes, how long does the whole operation take from the moment he puts the first egg in to the moment he takes the seventh egg out?

A4 The hobbits Frodo, Sam, Pippin and Merry have breakfast at different times. Each one takes a quarter of the porridge in the pan, thinking that the other three have not yet eaten. What fraction of the porridge is left after all four hobbits have had their breakfast?

A5 The diagram shows a tower consisting of three identical dice.

On these dice, each pair of opposite faces has a total of seven dots.

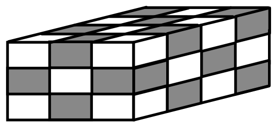
How many dots are there on the face on which the tower stands?



A6 The sizes in degrees of the interior angles of a pentagon are consecutive whole numbers. What is the size of the largest of these angles?

A7 A large cuboid is made from cuboids of equal size, coloured alternately black and white, as shown.

What fraction of the surface area of the large cuboid is black?



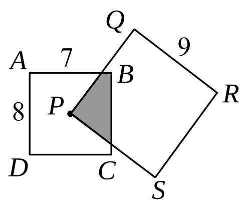
A8 Pegs numbered 1 to 50 are placed in order in a line with number 1 on the left. They are then knocked over one at a time following these rules:

- Of the pegs which are still standing, knock down alternate ones, starting with the first peg on the left.
- Each time you reach the end of the row, repeat the previous rule.

What is the number of the last peg to be knocked over?

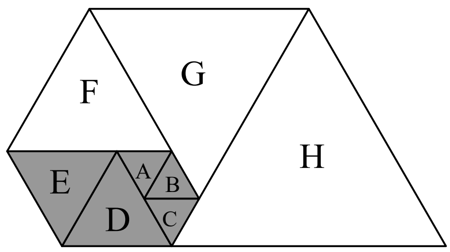
A9 The diagram shows squares $ABCD$ and $PQRS$ of side length 8 units and 9 units respectively. Point P is the centre of square $ABCD$; PQ intersects AB at a point 7 units from A.

What is the perimeter of the shaded region?



A10 The diagram shows a spiral of equilateral triangles. After the first five triangles A, B, C, D, E (shown shaded), the next triangle is always placed alongside two others: the one placed immediately before and one placed earlier. The smallest triangles have sides of length 1 unit.

What is the length of the sides of the fifteenth triangle?

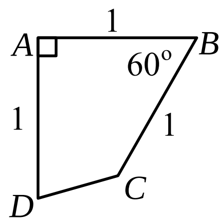


Section B

B1 Find four integers whose sum is 400 and such that the first integer is equal to twice the second integer, three times the third integer and four times the fourth integer.

B2 The diagram shows a quadrilateral $ABCD$ in which AB, BC and AD are all of length 1 unit, $\angle BAD$ is a right angle and $\angle ABC$ is 60°.

Prove that $\angle BDC = 2 \times \angle DBC$.

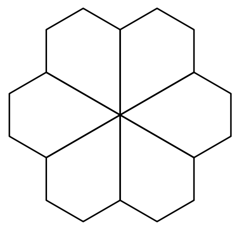


B3 (a) Yesterday evening, my journey home took 25% longer than usual.
By what percentage was my average speed reduced compared to normal?

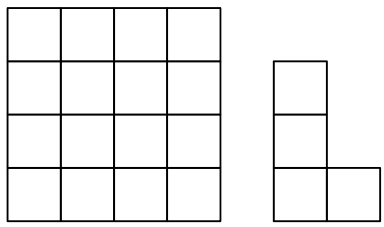
(b) By what percentage would I need to increase my usual average speed in order for the journey to take 20% less time than usual?

B4 Find a rule which predicts exactly when five consecutive integers have sum divisible by 15.

B5 A window is constructed of six identical panes of glass. Each pane is a pentagon with two adjacent sides of length two units.
The other three sides of each pentagon, which are on the perimeter of the window, form half of the boundary of a regular hexagon
Calculate the exact area of glass in the window.



B6 We want to colour red some of the cells in the 4×4 grid shown so that wherever the L-shaped piece is placed on the grid it covers at least one red cell. The L-shaped piece may only cover complete cells, may be rotated, but may not be turned over and may not extend beyond the grid.



(a) Show that it is possible to achieve this by colouring exactly four cells red.
(b) Show that it is impossible to achieve this by colouring fewer than four cells red.

UK Junior Mathematical Olympiad 2008

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Tuesday 17th June 2008

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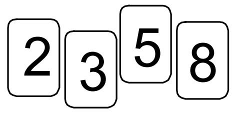
Section A

A1 In how many ways is it possible to place side by side two of the cards shown to form a two-digit prime number?



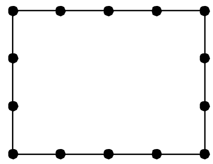
A2 Tony wants to form a square with perimeter 12 cm by folding a rectangle in half and then in half again. What is the maximum possible perimeter of the original rectangle?

A3 Given that $\frac{1}{2} + \frac{1}{3} + \frac{1}{12} + \frac{1}{18} + \frac{1}{x} = 1$, what is the value of x ?

A4 How many three-digit numbers have the product of their digits equal to 6?

A5 A 3 by 4 rectangle has 14 points equally spaced around its four sides, as shown.

In how many ways is it possible to join two of the points by a straight line so that the rectangle is divided into two parts which have areas in the ratio 1 : 3?



A6 How many positive square numbers are factors of 1600?

A7 In a *Magic Square*, the sum of the three numbers in each row, each column and each of the two main diagonals is the same.

What is the value of x in the partially completed magic square shown?

		6
x	4	5

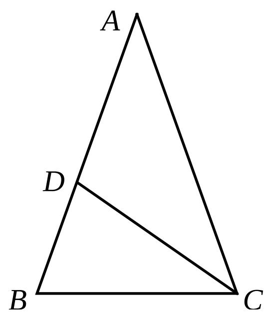
A8 Granny shares a packet of sweets between her four granddaughters. The girls, Clarrie, Lizzie, Annie and Danni, always in that order, each take 8 sweets in turn, over and over again until, finally, there are some sweets left for Danni, but there are fewer than 8. Danni takes all the sweets that are left. The other three girls then give Danni some of their sweets so that all four girls have the same number of sweets.

How many sweets does each of the other three granddaughters give to Danni?

A9 In the diagram, CD is the bisector of angle ACB.

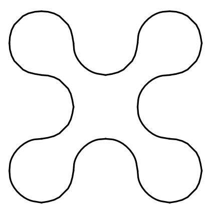
Also, $BC = CD$ and $AB = AC$.

What is the size of angle CDA?



A10 The perimeter of the shape shown on the right is made from 20 quarter-circles, each with radius 2 cm.

What is the area of the shape?



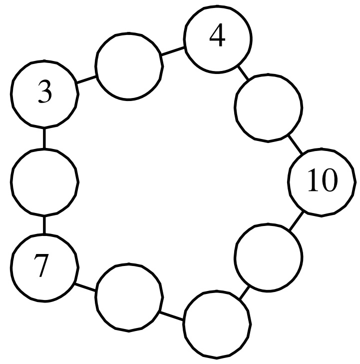
Section B

B1 Tamsin has a selection of cubical boxes whose internal dimensions are whole numbers of centimetres, that is, $1 \text{ cm} \times 1 \text{ cm} \times 1 \text{ cm}$, $2 \text{ cm} \times 2 \text{ cm} \times 2 \text{ cm}$, and so on.

What are the dimensions of the smallest of these boxes in which Tamsin could fit ten rectangular blocks each measuring $3 \text{ cm} \times 2 \text{ cm} \times 1 \text{ cm}$ without the blocks extending outside the box?

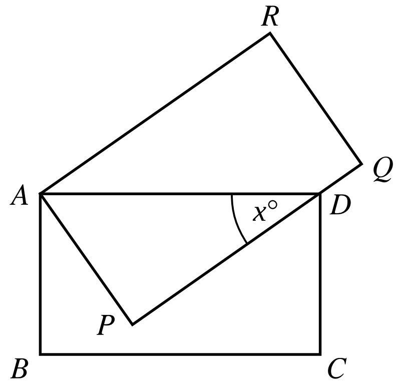
B2 Each of the numbers from 1 to 10 is to be placed in the circles so that the sum of each line of three numbers is equal to T. Four numbers have already been entered.

Find all the possible values of T.



B3 In the diagram $ABCD$ and $APQR$ are congruent rectangles. The side PQ passes through the point D and $\angle PDA = x°$.

Find an expression for $\angle DRQ$ in terms of x.

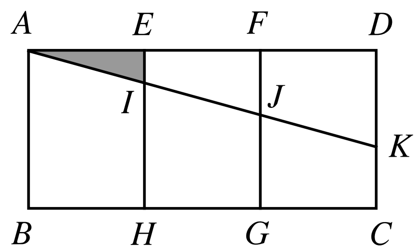


B4 For each positive two-digit number, Jack subtracts the units digit from the tens digit; for example, the number 34 gives $3 - 4 = -1$.

What is the sum of all his results?

B5 In the diagram, the rectangle $ABCD$ is divided into three congruent rectangles. The line segment JK divides $CDFG$ into two parts of equal area.

What is the area of triangle AEI as a fraction of the area of $ABCD$?



B6 In a sequence of positive integers, each term is larger than the previous term. Also, after the first two terms, each term is the sum of the previous two terms.

The eighth term of the sequence is 390. What is the ninth term?

UK Junior Mathematical Olympiad 2009

Organised by The United Kingdom Mathematics Trust

Tuesday 16th June 2009

RULES AND GUIDELINES :
READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING

1. Time allowed: 2 hours.
2. **The use of calculators, measuring instruments and squared paper is forbidden.**
3. All candidates must be in *School Year 8 or below* (England and Wales), *S2 or below* (Scotland), *School Year 9 or below* (Northern Ireland).
4. For questions in Section A *only the answer is required*. Enter each answer neatly in the relevant box on the Front Sheet. Do not hand in rough work. Write in blue or black pen or pencil.

   For questions in Section B you must give *full written solutions*, including clear mathematical explanations as to why your method is correct.

   Solutions must be written neatly on A4 paper, starting each question on a fresh sheet.

   Sheets must be STAPLED together in the top left corner with the Front Sheet on top.

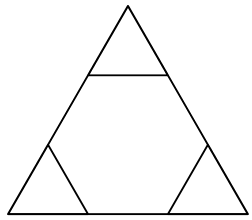
   Do not hand in rough work.
5. Questions A1-A10 are relatively short questions. Try to complete Section A within the first 45 minutes so as to allow well over an hour for Section B.
6. Questions B1-B6 are longer questions requiring *full written solutions*.
   This means that each answer must be accompanied by clear explanations and proofs.
   Work in rough first, then set out your final solution with clear explanations of each step.
7. These problems are meant to be challenging! Do not hurry. Try the earlier questions in each section first (they tend to be easier). Try to finish whole questions even if you can't do many. A good candidate will have done most of Section A and given solutions to at least two questions in Section B.
8. Answers must be FULLY SIMPLIFIED, and EXACT using symbols like π, fractions, or square roots if appropriate, but NOT decimal approximations.

DO NOT OPEN THE PAPER UNTIL INSTRUCTED BY THE INVIGILATOR TO DO SO!

Section A

A1 What is the value of $200^2 + 9^2$?

A2 The diagram shows a regular hexagon inside an equilateral triangle. The area of the larger triangle is $60\ \text{cm}^2$. What is the area of the hexagon?



A3 The positive whole numbers a, b and c are all different and $a^2 + b^2 + c^2 = 121$. What is the value of $a + b + c$?

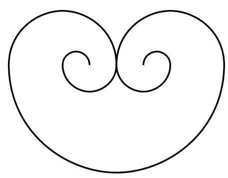
A4 The sum of three numbers is 2009. The sum of the first two numbers is 1004 and the sum of the last two is 1005. What is the product of all three numbers?

A5 Andrea's petrol tank holds up to 44 litres of fuel. She goes to the garage when her tank is a quarter full and puts more petrol in the tank until it is two-thirds full. How many litres of petrol does she put in the tank?

A6 The shorter sides of a right-angled isosceles triangle are each 10 cm long. The triangle is folded in half along its line of symmetry to form a smaller triangle. How much longer is the perimeter of the larger triangle than that of the smaller?

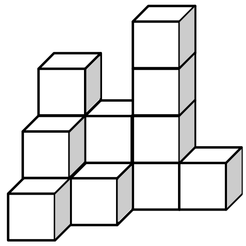
A7 Dean runs on a treadmill for thirty minutes. To keep his mind active as well as his legs, he works out what fraction of the total time has passed at each half minute and minute from the start. How many of the results of his calculations can be expressed in the form $\frac{1}{n}$, where n is an integer greater than 1?

A8 The diagram shows a curve made from seven semicircular arcs, the radius of each of which is 1 cm, 2 cm, 4 cm or 8 cm. What is the length of the curve?



A9 A book has 89 pages, but the page numbers are printed incorrectly. Every third page number has been omitted, so that the pages are numbered 1, 2, 4, 5, 7, 8, … and so on. What is the number on the last printed page?

A10 Gill piles up fourteen bricks into the shape shown in the diagram. Each brick is a cube of side 10 cm and, from the second layer upwards, sits exactly on top of the brick below.
Including the base, what is the surface area of Gill's construction?



Section B

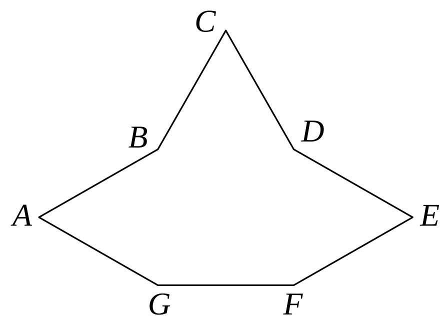
Your solutions to Section B will have a major effect on your JMO results. Concentrate on one or two questions first and then **write out full solutions** (not just brief 'answers').

B1 In 2007 Alphonse grew twice the number of grapes that Pierre did. In 2008 Pierre grew twice the number of grapes that Alphonse did. Over the two years Alphonse grew 49 000 grapes, which was 7600 less than Pierre. How many grapes did Alphonse grow in 2007?

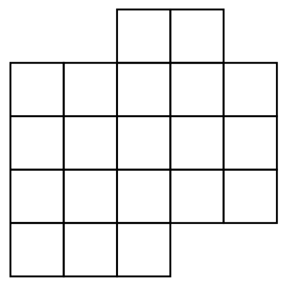
B2 *ABCD* is a square. The point *E* is outside the square so that *CDE* is an equilateral triangle. Find angle *BED*.

B3 Tom left a motorway service station and travelled towards Glasgow at a steady speed of 60 mph. Tim left the same service station 10 minutes after Tom and travelled in the same direction at a steady speed, overtaking Tom after a further 1 hour 40 minutes. At what speed did Tim travel?

B4 The diagram shows a polygon *ABCDEFG*, in which $FG = 6$ and $GA = AB = BC = CD = DE = EF$. Also *BDFG* is a square. The area of the whole polygon is exactly twice the area of *BDFG*. Find the length of the perimeter of the polygon.



B5 An ant wishes to make a circuit of the board shown, visiting each square exactly once and returning to the starting square. At each step the ant moves to an adjacent square across an edge. Two circuits are considered to be the same if the first follows the same path as the second but either starts at a different square or follows the same path in reverse. How many such circuits are possible?



B6 I want to choose a list of *n* different numbers from the first 20 positive integers so that no two of my numbers differ by 5. What is the largest value of *n* for which this is possible? How many different lists are there with this many numbers?

UK Junior Mathematical Olympiad 2005

Organised by The United Kingdom Mathematics Trust

Tuesday 14th June 2005

RULES AND GUIDELINES :
READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING

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2. **The use of calculators *and measuring instruments* is forbidden.**

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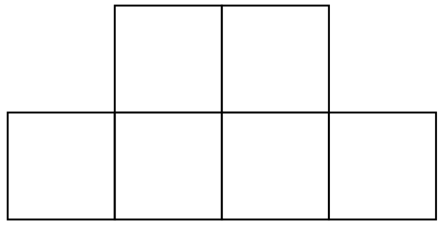
Section A

A1 How many seconds are there in one fortieth of an hour?

A2 The diagram shows a shape made from six squares, each of side 1cm.

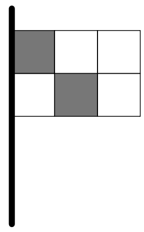
Four copies of the shape are placed together (without leaving any holes or having any overlaps) to form a rectangle.

What is the perimeter of the rectangle?



A3 Three different integers have a sum of 1 and a product of 36. What are they?

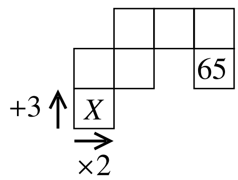
A4 A picture of a flag is to be completed by shading two squares which do not share an edge. The diagram shows one way in which this can be done.
How many different possible completed pictures are there (including the one shown)?



A5 In this puzzle, when you move up one square you **add 3**, when you move down one square you **subtract 3** and when you move to the right one square you **multiply by 2**.

The last square contains the number 65.

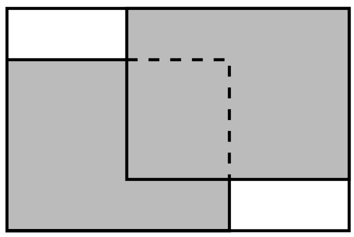
What number is in the square marked X ?



A6 Charlie's factory makes crème eggs and caramel eggs. The crème eggs are produced by a machine at the rate of 30 per minute, while the caramel eggs are produced by a different machine at the rate of 40 per minute. On a day when these two machines were in operation for a combined time of 18 hours, 36 000 eggs were produced in total. For how many hours was the crème egg machine in use?

A7 A sheet of paper is exactly the same size as a rectangular table top. The paper is cut in half and the two halves are placed on the table as shown.

What is the ratio of the area of table left uncovered (white) to the area which is covered twice?



A8 A large container holds 14 litres of a solution which is 25% antifreeze, the remainder being water. How many litres of antifreeze must be added to the container to make a solution which is 30% antifreeze?

A9 Colin has a collection of more than 24 coins. When he puts the coins in piles of 6, there are 3 coins remaining. When he puts the coins in piles of 8, there are 7 coins remaining. How many coins remain when he puts the coins in piles of 24?

A10 A closed rectangular box is a 'double cube', in which the top and bottom are squares, and the height is twice the width. The greatest distance between any two points of this box is 9 cm. What is the total surface area of the box?

Section B

Your solutions to Section B will have a major effect on the JMO results. Concentrate on one or two questions first and then **write out full solutions** (not just brief ‘answers’).

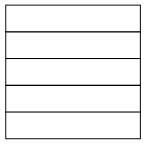
B1 The first three terms of a sequence are $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$. The fourth term is $\frac{1}{2} - \frac{1}{3} + \frac{1}{4}$; henceforth, each new term is calculated by taking the previous term, subtracting the term before that, and then adding the term before that.

(i) Write down the first six terms of the sequence, giving your answers as simplified fractions.

(ii) Find the 10th term and the 100th term, and explain why they have to be what you claim.

B2 The diagram shows a square which has been divided into five congruent rectangles. The perimeter of each rectangle is 51 cm.

What is the perimeter of the square?



B3

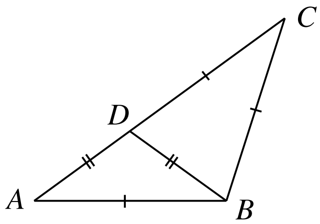
	175										70	

The diagram above is to be completed so that each box contains a whole number, the total of the numbers in the thirteen boxes is 2005 and the sum of the numbers in any three consecutive boxes is always the same.

In how many different ways is it possible to complete the diagram in this way?

B4 In this figure ADC is a straight line and $AB = BC = CD$. Also, $DA = DB$.

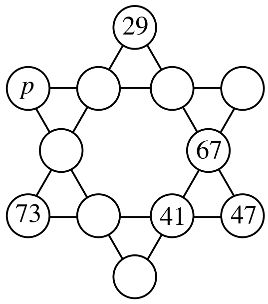
Find the size of $\angle BAC$.



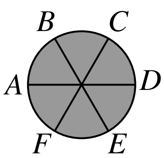
B5 In a magic hexagram, the numbers in every line of four circles have the same total. The diagram shows a magic hexagram which uses twelve different prime numbers.

Five numbers, including the smallest and the largest of the twelve primes, are shown.

Find the value of p, explaining the steps in your reasoning.



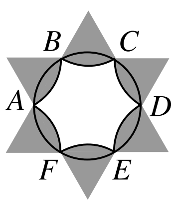
B6



Points A, B, C, D, E and F are equally spaced around a circle of radius 1. The circle is divided into six sectors as shown on the left.

The six sectors are then rearranged so that A, B, C, D, E and F lie on a new circle, also of radius 1, as shown on the right with the sectors pointing outwards.

Find the area of the curvy *unshaded* region.



UK Junior Mathematical Olympiad 2005 Solutions

A1 **90** One fortieth of an hour is one and a half minutes, that is 90 seconds.

A2 **20cm** When the four shapes are placed together, they will form a rectangle measuring 6cm by 4cm.

A3 **6, –3 , –2** As the sum of the integers is 1, it is clear that at least one of them is negative. Their product is positive so it may be deduced that exactly two of the integers are negative. Now we need to find three factors of 36 such that the largest is 1 greater than the sum of the other two. These are 6, 3 and 2.

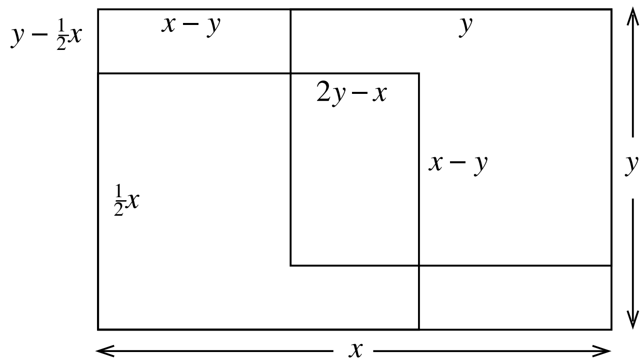
A4 **8** Number the squares 1 to 6 as shown. The possible pairings are 1 and 4, 1 and 5, 1 and 6, 2 and 3, 2 and 5, 2 and 6, 3 and 6, 4 and 5.

1	3	5
2	4	6

A5 **4** Working backwards from 65, we deduce that the numbers in the squares are 68, 34, 17, 14, 7, 4 respectively.

A6 **12** Let the crème egg machine be in use for x hours. Then $1800x + 2400(18 - x) = 36\,000$, that is $3x + 4(18 - x) = 60$, that is $72 - x = 60$.

A7 **1:1** Let the sheet of paper have length x and width y. Then the uncovered area consists of two congruent rectangles of length $x - y$ and width $y - \frac{1}{2}x$. So the uncovered area is $2(x - y)(y - \frac{1}{2}x)$, that is $(x - y)(2y - x)$.



The area covered twice is a rectangle of length $y - (x - y)$, that is $2y - x$, and width $\frac{1}{2}x - (y - \frac{1}{2}x)$, that is $(x - y)$. So the area covered twice is also $(x - y)(2y - x)$.

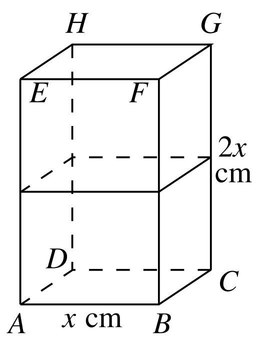
A8 **1** Let the extra volume of antifreeze required be x litres. Initially, there are 3.5 litres of antifreeze in the solution. So $3.5 + x = 0.3(14 + x)$, that is $35 + 10x = 42 + 3x$, so $x = 1$.

A9 **15** The number of coins in Colin's collection is 3 more than a multiple of 6 and also 7 more than a multiple of 8. The smallest number which satisfies both conditions is 15. The lowest common multiple of 6 and 8 is 24, so the conditions will also be met by numbers which exceed 15 by a multiple of 24, that is 39, 63, 87, etc. So when Colin puts his coins in piles of 24, 15 remain.

A10 **135cm²** Let the square base, $ABCD$, of the box be of side x cm. Then by Pythagoras' Theorem: $AC^2 = AB^2 + BC^2 = 2x^2$.

The greatest distance between any two points of the box is equal to the distance between opposite corners such as A and G. Applying Pythagoras' Theorem to triangle ACG: $AG^2 = AC^2 + CG^2 = 2x^2 + (2x)^2 = 6x^2$. So $6x^2 = 9^2 = 81$.

The total surface area, in cm², of the box $= 2 \times x^2 + 4 \times 2x^2 = 10x^2 = 10 \times \frac{81}{6} = 135$.



B1 (i) The fourth term = $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} = \frac{5}{12}$. The fifth term = $\frac{5}{12} - \frac{1}{2} + \frac{1}{3} = \frac{1}{4}$. The sixth term = $\frac{1}{4} - \frac{5}{12} + \frac{1}{2} = \frac{1}{3}$. So the first six terms are $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{5}{12}, \frac{1}{4}, \frac{1}{3}$.

(ii) The seventh term = $\frac{1}{3} - \frac{1}{4} + \frac{5}{12} = \frac{1}{2}$. So the fifth, sixth and seventh terms equal the first three terms respectively and this means that the sequence will repeat itself every four terms since each term depends upon the previous three terms only.

So the tenth term equals the second term, $\frac{1}{3}$, and the hundredth term equals the fourth term, $\frac{5}{12}$.

B2 Let the length of the side of the square be $5x$ cm. Then each of the congruent rectangles has length $5x$ cm and width x cm, giving a perimeter of $12x$ cm. So $12x = 51$, that is $x = 4\frac{1}{4}$. The perimeter of the square = $20x$ cm, i.e. 85 cm.

B3

x	175	y									70	

Let the numbers in the first and third boxes be x and y respectively. Then the sum of the numbers in the first three boxes is $x + 175 + y$. However, this must equal the sum of the numbers in the second, third and fourth boxes and as the first two of these are 175 and y respectively, the number in the fourth box is x. Similarly, the sum of the numbers in the third, fourth and fifth boxes is $x + 175 + y$ and as the first two of these are y and x respectively, the number in the fifth box is 175. This argument may be continued to show that if the sum of the numbers in any three consecutive boxes is the same then the sequence of numbers must be x, 175, y, x, 175, y, x, 175, y, … and the required condition may be met in exactly one way or not at all.

As the number in the twelfth box is 70, we may deduce that y is 70. So the numbers in the thirteen boxes are x, 175, 70, x, 175, 70, x, 175, 70, x, 175, 70, x respectively and it remains to test if the value of x which makes the total of these numbers equal to 2005 is a whole number.

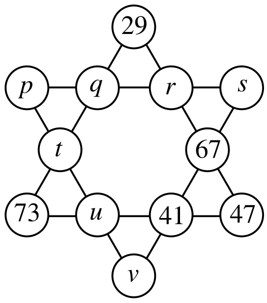
We require that $5x + 980 = 2005$, giving $x = 205$, so the diagram may be completed in exactly one way.

B4 Let $\angle BAC = x°$. Then $\angle BCA = x°$ since $AB = BC$. Also, $\angle ABD = x°$, since $DA = DB$. Furthermore, $\angle CDB = \angle DAB + \angle DBA = 2x°$ (exterior angle of a triangle).

As $BC = CD$, $\angle CBD = \angle CDB = 2x°$.

Triangle BCD has interior angles of $2x°$, $2x°$ and $x°$ so $5x = 180$ (angle sum of triangle), and so $\angle BAC = 36°$.

B5 The twelve prime numbers from 29 to 73 inclusive are 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73.



Let q, r, s, t, u, v represent the missing numbers in the circles shown.

Consider the two rows of four circles with one missing number each:

$73 + u + 41 + 47 = 47 + 67 + r + 29$ so $u = r - 18$. This means either that $r = 71$, $u = 53$ and each row totals 214 or $r = 61$, $u = 43$ and each row totals 204.

If the former is true, then the numbers still available are 31, 37, 43, 59, 61 and we require that $s + v = 214 - (41 + 67) = 106$. This is impossible.

If the latter is true, then the numbers still available are 31, 37, 53, 59, 71 and we require that $s + v = 204 - (41 + 67) = 96$. This is satisfied when s and v are 37 and 59 in some order.

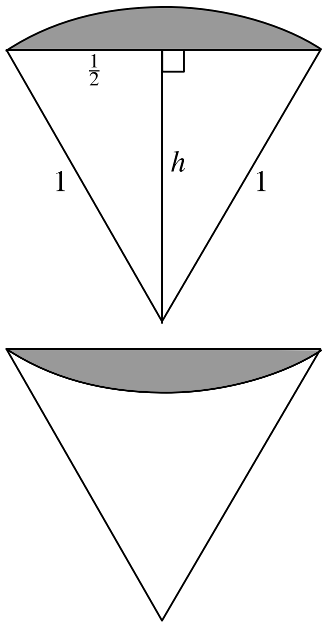
The numbers still available are 31, 53, 71.

If $s = 37$ and $v = 59$ then $p + q = 204 - (61 + 37) = 106$. This is impossible.

If $s = 59$ and $v = 37$ then $p + q = 204 - (61 + 59) = 84$. This is satisfied when p and q are 31 and 53 in some order. So t is 71 and we may deduce that $p = 204 - (71 + 43 + 37) = 53$.

(We may check that $q = 31$ does give a total of 204 for the rows containing 53, 31, 61, 59 and 29, 31, 71, 73 respectively.)

B6 The symmetry of the figure means that the required area may be divided into six equal parts. The area of each of these parts is the area an equilateral triangle of side 1 minus the area of the shaded segment shown in the diagram.



Let the height of the equilateral triangle be h. Then, using Pythagoras' Theorem: $h^2 + \frac{1}{2}^2 = 1^2$ so $h = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$ and the area of the triangle is $\frac{1}{2} \times \frac{1}{2}\sqrt{3} = \frac{1}{4}\sqrt{3}$.

Now the sector area is $\frac{1}{6} \times \pi \times 1^2$ so the area of the shaded segment is $\frac{1}{6}\pi - \frac{1}{4}\sqrt{3}$. Hence the area of the curvy unshaded region is .

$$6\left[\frac{\sqrt{3}}{4} - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)\right] = 3\sqrt{3} - \pi.$$

UK Junior Mathematical Olympiad 2006 Solutions

A1 **2006** $1 + 2 \times (3 + 4^5) + 6 + 7 - 8 \times 9 + 10 = 1 + 2 \times 1027 + 6 + 7 - 72 + 10$
$= 1 + 2054 - 49 = 2006$.

A2 **27cm³** Let the length of each edge of the cube be x cm. Then $14x = 42$, so $x = 3$.

A3 **792** The combinations of three digits which have a product of 36 are (1, 4, 9), (1, 6, 6), (2, 2, 9), (2, 3, 6), (3, 3, 4). So the greatest and smallest three-digit numbers for which the product of the digits equals 36 are 941 and 149 respectively.

A4 **11** If all three regions are painted the same colour, then three different painted triangles are possible. If exactly two regions are the same colour, there are three choices for that colour and, for each choice, the remaining region can be coloured two ways, giving six possibilities in all. If all three regions are different colours, there are only two possible painted triangles. Going clockwise round the triangle one will be red, yellow then blue and the other will be red, blue and then yellow. Any other painted triangle with all three colours used will be a rotation of one of these and so not counted as different. Hence a total of 11 different painted triangles may be made.

A5 **120** The balloon seller sells 1/3 of his balloons to boys and 1/5 of them to girls. The difference between these two fractions is 2/15, so he sells 6/15, that is 2/5, of the balloons to adults. The total number of balloons sold, therefore, is 14/15 of the number he started with. This means that the 8 balloons which remain represent 1/15 of the original number of balloons.

A6 **10°** The star is a 12-sided polygon, so the sum of its interior angles $= (12 - 2) \times 180°$ $= 1800°$. Each of its reflex angles is $360° - 70°$, that is $290°$. So each of the angles marked '$\circ$' $= (1800° - 6 \times 290°) \div 6 = 10°$.

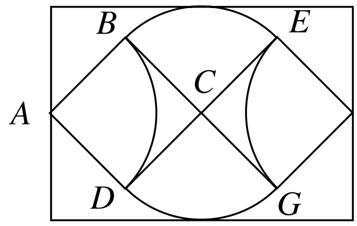
A7 **£7** The only factors of 781 are 1, 11, 71 and 781 so we can conclude that Tom is 11 years old and his grandfather is 71 years old. Last year they were 10 and 70 years old respectively, so Tom received 700p.

A8 **4** The maximum sum for any row or column is $7 + 8 + 9 = 24$, so these three numbers in some order make up row 3. Given that, the maximum sum for any column is $9 + 6 + 5 = 20$, so those three numbers in some order make up column three. Hence $i = 9$. The largest total for any other row is $6 + 4 + 3 = 13$, so those three numbers in some order make up row 2. Hence $f = 6$. The largest total for another column is $8 + 4 + 2 = 14$, so those three numbers in some order make up column 2. Hence $e = 4$.

a	b	c	8
d	e	f	13
g	h	i	24
11	14	20	

A9 **23** Three different prime numbers, one of which is 2, have a sum which is even, so we may conclude that 2 does not appear in either set of three prime numbers. The next prime numbers in ascending order are 3, 5, 7, 11 so the required number must be at least $3 + 5 + 11$, that is 19. However, this is the only way of writing 19 as the sum of three different primes. The smallest prime greater than 19 is 23, which equals $3 + 7 + 13$ and also equals $5 + 7 + 11$. So 23 is the smallest prime which may be written as the sum of three different primes in two different ways.

A10 $\sqrt{2}$ **m** The radius of each of the quarter circles is 50cm. So by symmetry we see that *ABCD* and *CEFG* are both squares of side 50cm.



From Pythagoras' Theorem:
$AC^2 = (50^2 + 50^2)\text{ cm}^2 = 5000\text{ cm}^2$.
So $AC = \sqrt{5000}\text{ cm} = 50\sqrt{2}\text{ cm}$.
Similarly, $CF = 50\sqrt{2}$ cm, so
$AF = 100\sqrt{2}\text{ cm} = \sqrt{2}\text{ m}$.

B1 Let the number of 10p and 20p coins which Jenny has be x and y respectively. Then the number of 50p coins she has is $(20 - x - y)$.

So $10x + 20y + 50(20 - x - y) = 500$, that is $x + 2y + 5(20 - x - y) = 50$.

So x and y are positive integers which satisfy the equation $4x + 3y = 50$.

The possibilities are $x = 11, y = 2$; $x = 8, y = 6$; $x = 5, y = 10$; $x = 2, y = 14$.

The number of 50p coins corresponding to these are 7, 6, 5, 4 respectively and as Jenny has more 50p coins than 10p coins, we may conclude that she has two 10p coins, fourteen 20p coins and four 50p coins.

B2 For the final number in the chain to be 6, the penultimate number must be 16, 23, 32 or 61.

Any of these could be the first number in a chain of length 2, but 16 or 32 could be the penultimate number in a longer chain. However, 23 and 61 are prime, so cannot be the product of the digits of a previous number.

If the penultimate number is 16, the previous number must be 28, 44 or 82. Of these, only 28 is the product of two single digits and would be the next number in a chain after 47 or 74. In either case, 28 would be the second number in the chain since 47 is prime and 74 cannot be written as the product of two single digits.

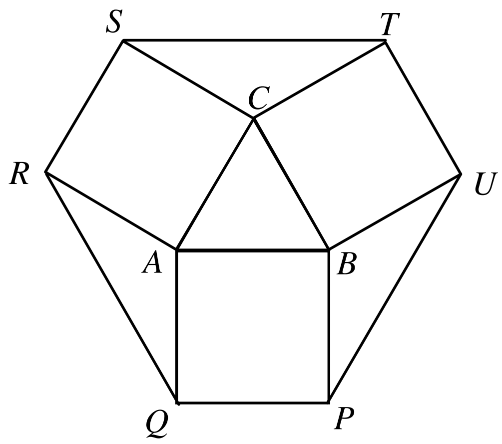
If the penultimate number is 32, the previous number must be 48 or 84. Of these, only 48 is the product of two single digits and would be the next number in a chain after 68 or 86. In either case, 32 would be the second number in the chain since neither 68 nor 86 may be written as the product of two single digits.

So the possible two-digit first numbers for a chain are, in ascending order: 16, 23, 28, 32, 44, 47, 48, 61, 68, 74, 82, 84, 86.

B3 Let $\angle XBC = x°$. Then $\angle BXY = 180° - x°$ as these two angles are angles inside parallel lines *XY* and *BC*.

Also, $\angle AXY = 7x°$ (given), so $\angle AXB = [360° - (180° - x°) - 7x°] = 180° - 6x°$ as angles at a point total 360°. Hence in isosceles triangle *ABX*, angles *XBA* and *XAB* total $[180° - (180° - 6x°)]$, that is $6x°$. So, since $AX = BX$, $\angle XBA = \angle XAB = 3x°$. Furthermore, as *AX* bisects angle *BAC*, $\angle XAY = 3x°$. So in triangle *AXY*, $7x + 3x + 90 = 180$; hence $x = 9$. The size of angle *ABC*, therefore, is $4 \times 9°$, that is 36°.

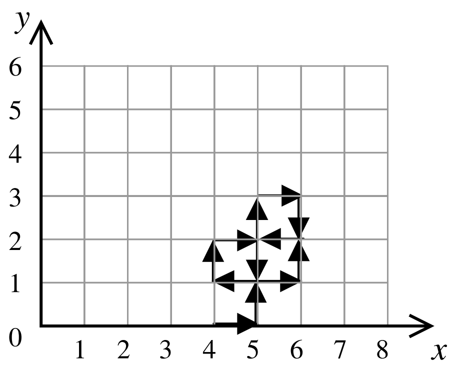
B4 Consider triangle STC: the angles meeting at C total 360°, so $\angle SCT = (360 - 60 - 90 - 90)°$, that is 120°. As $CS = CT = 2$, the perpendicular from C to TS divides triangle STC into two congruent triangles, each having interior angles of 30°, 60°, 90°. So these two triangles may be placed together to form an equilateral triangle of side 2; therefore triangle STC is equal in area to triangle ABC, as, by symmetry, are triangles AQR and BPU.



By Pythagoras' Theorem, the perpendicular height of triangle $ABC = \sqrt{2^2 - 1^2} = \sqrt{3}$, so the area of triangle ABC is $\frac{1}{2} \times 2 \times \sqrt{3}$, that is $\sqrt{3}$. Each of the squares $ABPQ$, $BCTU$, $CARS$ has area 4, so the area of hexagon $PQRSTU$ is $12 + 4\sqrt{3}$.

B5 (a) The diagram shows the points visited in the first 12 moves.

After (4, 0), the points visited, in order, are (5, 0), (5, 1), (4, 1), (4, 2), (5, 2), (5, 1), (6, 1), (6, 2), (5, 2), (5, 3), (6, 3), (6, 2). So it is at (6, 2) after 12 moves and is facing 'East'.

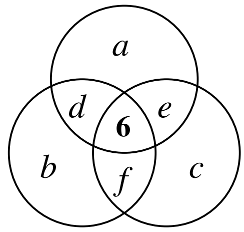


(b) Note that the effect of the first 6 moves is to move the bug 1 unit in both the positive x and y directions and to leave it facing 'East'. As the value of x has increased by 1 and the value of y has also increased by 1, the series of values which $(x - y)$ takes over the next 6 moves will be the same as those for the first 6 moves with the effect that the bug will once again move 1 unit in both the positive x and y directions and will face 'East' after this next series of 6 moves. This process will then continue for the next 6 moves and so on.

So, as has already been seen, the bug will be at the point (6, 2) after 12 moves. It will subsequently be at (7, 3) after 18 moves, at (8, 4) after 24 moves and so on. So it will be at (12, 8) after 48 moves.

As it is now facing 'East' it will be at (13, 8) after the 49th move and will then turn left and move to (13, 9), its position after 50 moves.

B6 Let the numbers inside the regions be a, b, c, d, e, f as shown.
Then: $a + d + e + 6 = T$; $b + d + f + 6 = T$; $c + e + f + 6 = T$.



Adding these equations gives $a + b + c + 2d + 2e + 2f + 18 = 3T$.
Now a, b, c, d, e, f are 1, 2, 3, 4, 5, 7 in some order, so $a + b + c + d + e + f = 22$.

Therefore $22 + d + e + f + 18 = 3T$, that is $40 + d + e + f = 3T$.

Now the minimum value of $d + e + f = 1 + 2 + 3 = 6$ and the maximum value $= 4 + 5 + 7 = 16$. So $46 \leqslant 3T \leqslant 56$. Since T is a positive integer, the only values it can have is 16, 17 or 18.

If $T = 16$, then $d + e + f = 8$ and the task may be completed with $a = 7, b = 4, c = 3, d = 1, e = 2, f = 5$.

If $T = 17$, then $d + e + f = 11$. However, $a + d + e + 6 = T = 17$, so $a + d + e = 11$. This requires a to equal f, which is impossible, so T cannot be 17.

If $T = 18$, then $d + e + f = 14$ and the task may be completed with $a = 5, b = 2, c = 1, d = 3, e = 4, f = 7$.

Thus the only possible values of T are 16 and 18.

UK Junior Mathematical Olympiad 2007 Solutions

A1 **1** $1^5 - 2^4 + 3^3 - 4^2 + 5^1 = 1 - 16 + 27 - 16 + 5 = 33 - 32 = 1$.

A2 **4** If '$7k$ minutes past nine' is the same time as '$8k$ minutes to ten' then $7k + 8k = 60$, so $k = 4$. (*The two times are 28 minutes past nine and 32 minutes to ten.*)

A3 **9 minutes** Charlie puts the seventh egg in the pan six minutes after he puts in the first egg. The seventh egg takes three minutes to cook, so he takes it out of the pan nine minutes after starting the whole operation.

A4 $\mathbf{\frac{81}{256}}$ After each hobbit eats his porridge, $\frac{3}{4}$ of what he started with remains. So after four have eaten, what remains is $\left(\frac{3}{4}\right)^4 = \frac{81}{256}$ of the original amount.

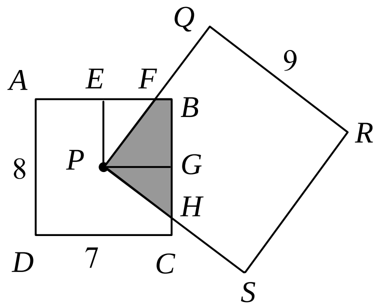
A5 **2** By comparing the dice at the top and bottom of the tower, it can be seen that the top face of the bottom die has five spots. So the face opposite, namely the face on which the tower stands, has two dots.

A6 **110°** The sum of the five interior angles of a pentagon is 540°, so the average size of these angles is 108°. As the sizes in degrees of the angles are consecutive whole numbers, they are 106°, 107°, 108°, 109°, 110°.

A7 $\mathbf{\frac{1}{2}}$ The visible end-face of the large cuboid consists of nine rectangles: five coloured white and four black. The face opposite this face also consists of nine rectangles: four coloured white and five black. So, between them, these two faces have equal numbers of white and black rectangles. Each of the other four faces of the large cuboid consists of twelve rectangles: six coloured white and six black. So the fraction of the surface area of the large cuboid which is coloured black is equal to the fraction which is coloured white, that is one half.

A8 **32** After the first pass, all the odd pegs have been knocked over and just the multiples of 2 remain standing. Likewise, after the next passes, first just the multiples of 4 remain, then those of 8, 16, 32. The final pass knocks down the only remaining peg, number 32.

A9 **18** Let the midpoints of *AB* and *BC* be *E* and *G* respectively and let *F* and *H* be the points shown.



Consider triangles *PEF* and *PGH*:
$\angle PEF = \angle PGH = 90°$; $\angle EPF = \angle GPH$, since both are equal to $90° - \angle FPG$;
$PE = PG$.

So the two triangles are congruent (AAS).

Applying Pythagoras' Theorem to triangle *PEF*: $PF^2 = PE^2 + EF^2 = 16 + 9 = 25$.

So *PF* has length 5 units.

As triangle *PGH* is congruent to triangle *PEF*, $GH = EF = 3$; $PH = PF = 5$.

So the perimeter of quadrilateral *PFBH* is $(5 + 1 + 7 + 5)$ units = 18 units.

(*Note that the area of overlap between the two squares remains constant when square PQRS is rotated about point P, but the perimeter of the overlapping region changes*.)

A10 **37** The lengths of the sides of the triangles in the figure are shown in the table below:

Triangle	A	B	C	D	E	F	G	H
Length of side	1	1	1	2	2	3	4	5

The ninth triangle, I, will be placed alongside triangles H and D, so its sides will be 7 units long.

The tenth triangle, J, will be placed alongside triangles I and E, so its sides will be 9 units long.

The eleventh triangle, K, will be placed alongside triangles J and F, so its sides will be 12 units long.

As the spiral continues, each new triangle is placed alongside the triangle placed immediately before it in the sequence and the triangle placed five turns earlier than the new triangle. So, the twelfth triangle, L, will be placed alongside triangles K and G, giving it sides of length 16 units. The lengths of the sides of the next three triangles to be placed are shown in the table below.

Triangle number	Placed alongside	Length of side
13 (M)	L and H	16 + 5 = 21
14 (N)	M and I	21 + 7 = 28
15 (O)	N and J	28 + 9 = 37.

B1 Let the first integer be x. Then the second, third and fourth integers are $\frac{x}{2}, \frac{x}{3}, \frac{x}{4}$ respectively.

Therefore $x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 400$, that is $\frac{25x}{12} = 400$. So $x = 192$ and the four integers, in order, are 192, 96, 64, 48.

(*Note that a simpler solution is obtained by letting the first integer be* $12x$.)

B2 Consider triangle ABC: it has two equal sides, AB and BC, and the angle between them is 60°, so it is equilateral. Therefore $\angle ACB = \angle CAB = 60°$ and AC has length 1 unit.

Now consider triangle ACD: $\angle CAD = \angle DAB - \angle CAB = 90° - 60° = 30°$. Also, $CA = DA = 1$ unit, so $\angle ACD = \angle ADC = 75°$.

Next consider triangle ABD. It is a right-angled isosceles triangle, therefore $\angle ABD = \angle ADB = 45°$.

Finally, consider triangle BCD: $\angle BDC = \angle ADC - \angle ADB = 75° - 45° = 30°$; $\angle DBC = \angle ABC - \angle ABD = 60° - 45° = 15°$. So $\angle BDC = 2 \times \angle DBC$.

B3 (a) Let my distance from work be d and the normal journey time be t. Then my normal average speed is $\frac{d}{t}$.

Yesterday, my journey time was 25% longer than usual, that is $t \times \frac{5}{4}$.

Therefore yesterday's average speed was $d \div \frac{5t}{4} = d \times \frac{4}{5t} = \frac{4d}{5t} = \frac{4}{5} \times \frac{d}{t}$.

So my average speed yesterday was 80% of its usual value; hence it was reduced by 20%.

(b) If the journey is to take 20% less time than usual, then the new journey time will need to be $\frac{4t}{5}$.

Therefore the average speed will need to be $d \div \frac{4t}{5} = d \times \frac{5}{4t} = \frac{5}{4} \times \frac{d}{t}$.

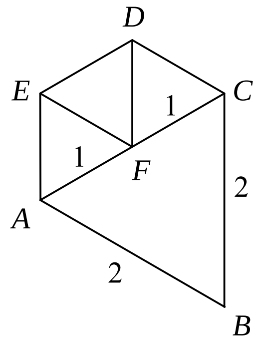
So my usual average speed will need to be increased by 25% of its normal value.

B4 Let the five consecutive integers be $x, x + 1, x + 2, x + 3$ and $x + 4$.

The sum of the numbers is $5x + 10 = 5(x + 2)$. This is a multiple of 15 if and only if 3 divides $x + 2$. So the sum of five consecutive integers is a multiple of 15 if and only if the third number is a multiple of 3.

(*A shorter proof is obtained by letting the consecutive integers be* $x - 2, x - 1, x, x + 1$ *and* $x + 2$. *Their sum is* $5x$, *which is clearly a multiple of* 5. *So it will be a multiple of* 15 *if and only if* x *is a multiple of* 3.)

B5 Let *ABCDE* be one of the pentagonal panes which make up the window and let *F* be the midpoint of *AC*.



As six equal angles meet at point *B*, $\angle ABC = 360° \div 6 = 60°$.

So triangle *ABC* has equal sides, *AB* and *BC*, and the angle between them is 60°; so it is equilateral. Hence *AC* has length 2.

As *F* is the midpoint of *AC*, $\angle AFB = 90°$, so, by Pythagoras' Theorem: $BF = \sqrt{AB^2 - AF^2} = \sqrt{4 - 1} = \sqrt{3}$.

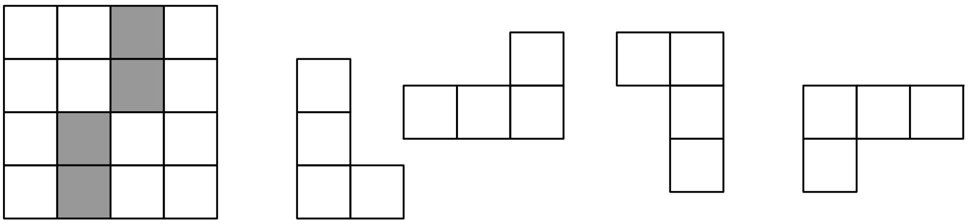
Therefore the area of triangle *ABC* is $\frac{1}{2} \times 2 \times \sqrt{3} = \sqrt{3}$.

As *ACDE* forms half of a regular hexagon, its area may be divided up into three congruent equilateral triangles, each of side 1. Consider one of these triangles: it is similar to triangle *ABC* with sides in the ratio 1:2, so its area is one quarter of the area of triangle *ABC*, that is $\frac{\sqrt{3}}{4}$.

Therefore the area of pentagon *ABCDE* is $\sqrt{3} + 3 \times \frac{\sqrt{3}}{4} = \frac{7\sqrt{3}}{4}$.

So the exact area of glass in the window is $6 \times \frac{7\sqrt{3}}{4} = \frac{21\sqrt{3}}{2}$.

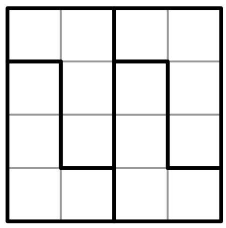
B6



(a) The diagrams show the four different orientations in which the L-shaped piece may be placed on the grid. It can be seen that when four cells of the grid are coloured as shown, it is not possible to place the L-shaped piece, whatever its orientation, on the grid without at least one of the coloured cells being covered.

So by colouring four cells red it is possible to ensure that wherever the L-shaped piece is placed on the grid it covers at least one red cell.

(b) The diagram shows how four copies of the L-shaped piece may be placed on the 4×4 grid without overlap. If fewer than four cells of the grid are coloured red then at least one of the four copies will have none of its cells coloured, so it will be possible to place the L-shaped piece on the grid without it covering at least one red cell. So if fewer than four cells are coloured red, it is impossible to ensure that wherever the L-shaped piece is placed on the grid it covers at least one red cell.



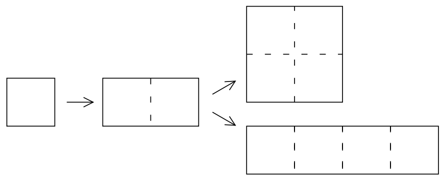
(*Please note that this is not the only diagram which may used show that the required task is impossible to achieve by colouring fewer than four cells red.*)

UK Junior Mathematical Olympiad 2008 Solutions

A1 3 A two-digit prime cannot end with a 2, 5 or 8. So we need only check 23, 53 and 83, each of which is prime.

A2 30 cm As the square has a perimeter of 12 cm, it must have a side length of 3cm. Unfolding the square once gives a 6 cm × 3 cm rectangle.

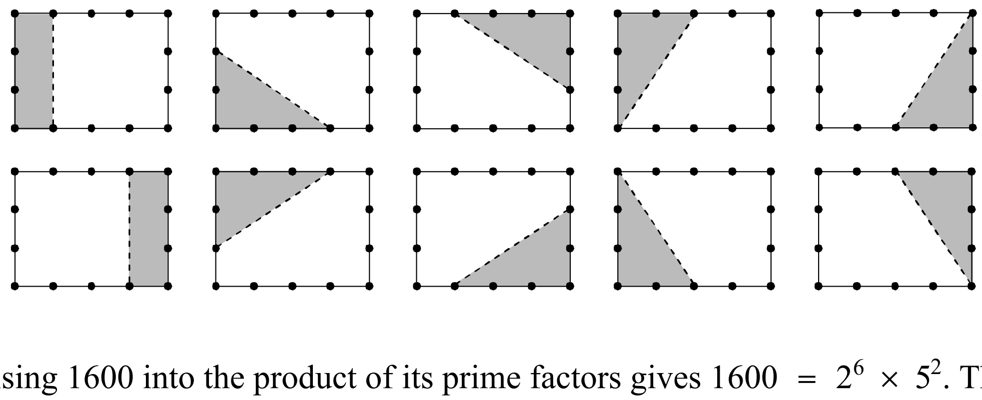
This rectangle can then be unfolded about either a long edge or a short edge resulting in a 6 cm × 6 cm square or a 12 cm × 3 cm rectangle. The perimeters of these are 24 cm and 30 cm respectively.



A3 $x = 36$ Rearranging the equation, we have $\frac{1}{x} = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{12} - \frac{1}{18} = \frac{36 - 18 - 12 - 3 - 2}{36} = \frac{1}{36}$, and so $x = 36$.

A4 9 The digits of a three-digit number whose product is 6 must be either two 1s and a 6, or a 1, 2 and 3. In the first case, there are three choices where to put the 6; in the other case, there are three choices for the first digit and then two choices for the second. Hence, altogether, there are 9 such numbers: 116, 161, 611 and 123, 132, 213, 231, 312, 321.

A5 10 The smaller area must be either a rectangle or a triangle of area 3. There are only two such rectangles, one at each end, as shown. The triangle needs to have base 2 and height 3 (or vice versa). There are two positions on each edge for the base. So we get eight triangles this way, as shown.



A6 8 Factorising 1600 into the product of its prime factors gives $1600 = 2^6 \times 5^2$. The factors of 1600 are either 1 or the numbers less than or equal to 1600 whose prime factors are only 2 or 5. Of the latter, the square numbers are those where the powers of 2 and of 5 are even. So the square numbers that are factors of 1600 are 1, 2^2, 2^4, 2^6, 5^2, $2^2 \times 5^2$, $2^4 \times 5^2$ and 1600 itself

[*Alternatively*: A factor of 1600 is a square number if and only if its square root is a factor of $\sqrt{1600} = 40$. There are eight factors of 40: 1, 2, 4, 5, 8, 10, 20 and 40.]

A7 **$x = 0$** As the bottom row and the diagonal running from 6 to x have the same sum, $x + 9$, the middle square must contain the number 3. Thus the remaining numbers in the top row must be $x + 2$ (from the second column) and $x + 1$ (from the diagonal). Considering the top row, we have $x + 1 + x + 2 + 6 = x + 9$, thus $x = 0$.

$x+1$	$x+2$	**6**
	3	
$\boldsymbol{x}$	**4**	**5**

[*Alternatively*: It is well known that the magic sum of a 3×3 magic square is three times the middle number. Once the middle number is known it follows that $x = 0$.]

A8 **1** After Annie has taken her last 8 sweets, let x be the number of sweets that Clarrie, Lizzie and Annie have and s be the number of sweets left for Danni, where $0 < s < 8$. After Danni takes the remaining s sweets, she has a total of $x - 8 + s$.

Let the three girls each give e sweets to Danni, so that

$$x - e = x - 8 + s + 3e$$

which gives

$$4e = 8 - s.$$

Because $0 < 4e < 8$, we have $e = 1$.

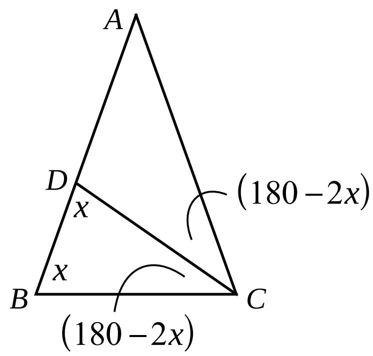
[*Alternatively*: Since the girls end up with equal numbers of sweets, the total number is a multiple of 4. They have all picked multiples of 8 except for the final number picked by Danni, and so she gets 4 sweets in her last turn. If each of the three other girls gives Danni 1 sweet, they all effectively gain 7 sweets in their last turn, whereas if they each gave more than 1 sweet, Danni would have more than them.]

A9 **108°** Triangle BCD is isosceles, so let $\angle DBC = \angle BDC = x°$.

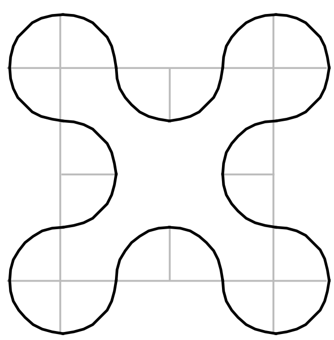
Then $\angle BCD = \angle ACD = (180 - 2x)°$.

Since triangle ABC is also isosceles, $\angle ACB = \angle ABC = x°$, and so $x = 180 - 2x + 180 - 2x$, and hence $x = 72$.

Thus $\angle CDA = (180 - x)° = 108°$.



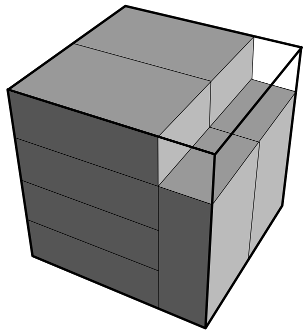
A10 **$(64 + 4\pi)$ cm^2** Adding the construction lines shown, we can see that the area of the shape
= the area of the square + area of 4 quarter-circles
$= (4 \times 2)^2 + \pi \times 2^2 = (64 + 4\pi)\,\text{cm}^2$.



B1 Tamsin has a selection of cubical boxes whose internal dimensions are whole numbers of centimetres, that is, 1 cm × 1 cm × 1 cm, 2 cm × 2 cm × 2 cm, and so on.

What are the dimensions of the smallest of these boxes in which Tamsin could fit ten rectangular blocks each measuring 3 cm × 2 cm × 1 cm without the blocks extending outside the box?

Solution



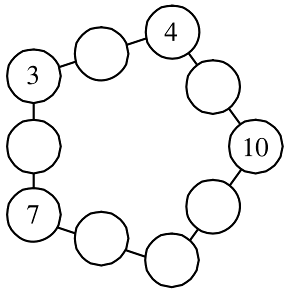
The total volume of 10 blocks, each 3 cm × 2 cm × 1 cm, is $10 \times 3 \times 2 \times 1 = 60$ cm^3.

The volumes of a 1 cm cube, a 2 cm cube and a 3 cm cube are 1 cm^3, 8 cm^3 and 27 cm^3 respectively so these cubes cannot contain the 10 blocks.

So the smallest *possible* cube is a 4 cm cube and the diagram on the right shows how such a cube can hold the 10 blocks.

B2 Each of the numbers from 1 to 10 is to be placed in the circles so that the sum of each line of three numbers is equal to T. Four numbers have already been entered.

Find all the possible values of T.



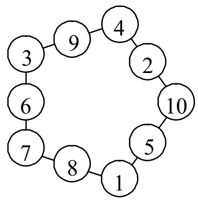
Solution

Let x be the number in the circle in the unfilled bottom corner of the pentagon.

All ten numbers are used once and only once and so the sum of the numbers in all ten circles is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$.

Now, consider the numbers as they appear in the five lines of three circles. When these five sets of numbers are added, the corner numbers (3, 4, 7, 10, x) will be included twice so the sum is

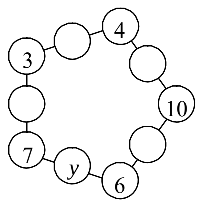
$$5T - 3 - 4 - 7 - 10 - x = 5T - x - 24.$$



Equating the two sums, we have $5T - x - 24 = 55$ which gives $5T = 79 + x$.

Since T is an integer, $5T$ is a multiple of 5, which means that x must be either 1 or 6.

If $x = 1$ then $T = 16$, while if $x = 6$, $T = 17$.



The first diagram shows that $x = 1$, $T = 16$ is a possibility, while in the second diagram, $x = 6$, $T = 17$ forces $y = 4$, repeating the 4 in the top corner.

Thus the only possible value of T is 16.

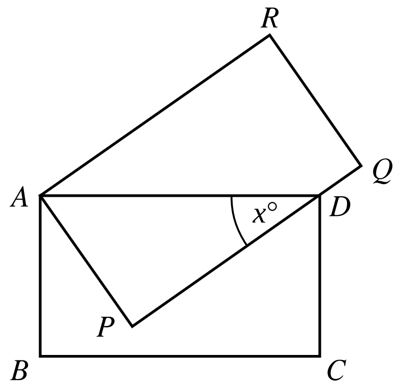
[*Alternatively*: Let the number in the circle between 4 and 10 be a. Then the total for each line T is $a + 14$, and so the number in the circle between 3 and 4 is $a + 7$. Yet this is between 1 and 10 and cannot be 7 or 10, and so $a = 1$ or $a = 2$.

If we try $a = 1$, we get $T = 15$, but there is no way of completing the other line including the 10, without reusing either the 3 or the 4.

If we try $a = 2$, we obtain the solution shown above, and hence $T = 16$ is the only possibility.]

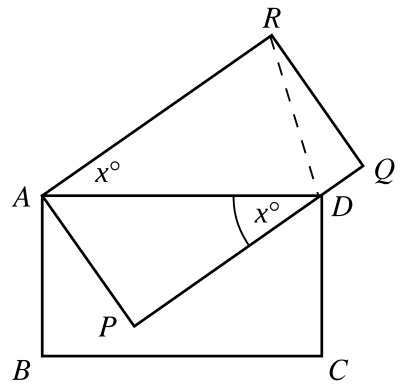
B3 In the diagram $ABCD$ and $APQR$ are congruent rectangles. The side PQ passes through the point D and $\angle PDA = x°$.

Find an expression for $\angle DRQ$ in terms of x.



Solution

Since $APQR$ is a rectangle, the lines AR and PQ are parallel.
Thus $\angle PDA = \angle DAR$ (alternate angles) so $\angle DAR = x°$.



Since $DA = RA$, triangle DAR is isosceles and so
$\angle ARD = \angle ADR = \frac{1}{2}(180 - x)° = \left(90 - \frac{1}{2}x\right)°$.

Hence, since $\angle ARQ = 90°$, $\angle DRQ = \frac{1}{2}x°$.

B4 For each positive two-digit number, Jack subtracts the units digit from the tens digit; for example, the number 34 gives $3 - 4 = -1$.

What is the sum of all his results?

Solution

The two-digit numbers run from 10 to 99.

We may categorise these numbers into the following four sets:

P = {the numbers ab where a and b are strictly positive and $a < b$}

Q = {the numbers ab where a and b are strictly positive and $b < a$}

R = {the palindromes aa}

S = {the numbers ab where $a > 0$ and $b = 0$}.

For numbers 'ab', the units digit subtracted from the tens digit is $a - b$.

The result $a - b$ from each of the numbers in set P can be matched with the result $b - a$ from each corresponding number in the set Q, giving a total of $(a - b) + (b - a) = 0$.

For each of the numbers in set R the result is $a - a$ which is 0.

Finally, when we consider all of the numbers in set S, the result is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$. So the total is 45.

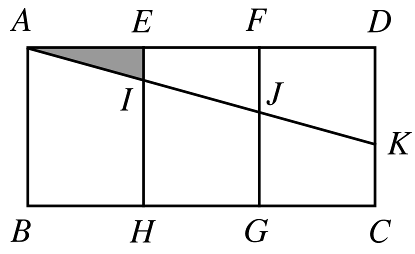
[*Alternatively*: The units digits for all the two-digit integers comprise the numbers 0, 1, ..., 9 each 9 times (coming after 1, 2, ..., 9).

The tens digits are 1, 2, ..., 8, 9, each coming 10 times (before 0, 1, ..., 9).

The difference is thus $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$.]

B5 In the diagram, the rectangle $ABCD$ is divided into three congruent rectangles. The line segment JK divides $CDFG$ into two parts of equal area.

What is the area of triangle AEI as a fraction of the area of $ABCD$?



Solution

Let a = the area of AEI.

We have $AF = 2 \times AE$ and $AD = 3 \times AE$. Also the triangles AEI, AFJ and ADK are similar since $\angle A$ is common and the angles at E, F and D are right-angles. Hence the area of $ADK = 9a$, the area of $AFJ = 4a$ and so the area of $DFJK = 9a - 4a = 5a$.

So the area of $DFGC = 2 \times 5a = 10a$ and the area of $ABCD = 3 \times 10a = 30a$.

Thus $\dfrac{\text{area } AEI}{\text{area } ABCD} = \dfrac{1}{30}$.

B6 In a sequence of positive integers, each term is larger than the previous term. Also, after the first two terms, each term is the sum of the previous two terms.

The eighth term of the sequence is 390. What is the ninth term?

Solution

Let the first two terms be the positive integers a and b. So the first nine terms of the sequence are:

$$a,\ b,\ a + b,\ a + 2b,\ 2a + 3b,\ 3a + 5b,\ 5a + 8b,\ 8a + 13b,\ 13a + 21b.$$

Thus $$8a + 13b = 390$$

giving $$8a = 390 - 13b = 13(30 - b). \qquad (*)$$

So $8a$ is a multiple of 13, and hence a itself is a multiple of 13.

Since $8a = 390 - 13b$, if $a \geqslant 26$ then $390 - 13b \geqslant 8 \times 26$, i.e. $13b \leqslant 390 - 208 = 182$ which means $b \leqslant 14 < a$ and the sequence would not be increasing.

So $a = 13$ and $13b = 390 - 13 \times 8$ giving $b = 22$.

Thus the ninth term is $13a + 21b = 13 \times 13 + 21 \times 22 = 169 + 462 = 631$.

[*Alternative ending*: From the starred equation, $30 - b$ is a multiple of 8, and hence could be 0, 8, 16 or 24. This means that b could be 30, 22, 14 or 6, from which the corresponding values of a are 0, 13, 26 and 39.

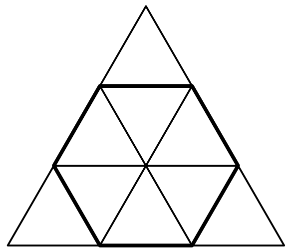
We know that $0 < a < b$, so the only possibility here is $a = 13$ and $b = 22$.

Thus the ninth term is $13a + 21b = 13 \times 13 + 21 \times 22 = 169 + 462 = 631$.]

UK Junior Mathematical Olympiad 2009 Solutions

A1 40081 $200^2 + 9^2 = 40000 + 81 = 40081$.

A2 40 cm² Since the hexagon is regular, it has interior angles of 120° and can be dissected into six congruent triangles. The small triangles have three angles of 60° and are therefore equilateral with side equal to that of the hexagon. The three triangles inside the large triangle but outside the hexagon are also equilateral with the same side length as the hexagon. So the area of the hexagon is $\frac{6}{9}\left(=\frac{2}{3}\right)$ of the area of the original triangle.



A3 17 It is clear that each of a, b and c must be less than or equal to 10. A brief inspection will show that the only combination of different square numbers which total 121 is $81 + 36 + 4$.

More formally, the problem can be analysed by considering the remainders after dividing the square numbers less than 121 (1, 4, 9, 16, 25, 36, 49, 64, 81 and 100) by three: the remainders are 1, 1, 0, 1, 1, 0, 1, 1, 0 and 1.

When 121 is divided by 3, the remainder is 1. Therefore $a^2 + b^2 + c^2$ must also leave a remainder of 1. Now we can deduce that two of the three squares must leave a remainder of 0 and so be multiples of 3. There are three square numbers below 121 which are multiples of three: 9, 36 and 81. Checking these, we see that 81 and 36 are the only pair to have a sum which differs from 121 by a perfect square, namely 4. So $a + b + c = 9 + 6 + 2 = 17$.

A4 0 The sum of the first two numbers and the last two numbers is $1004 + 1005 = 2009$. This counts the middle number twice. But the sum of all three numbers is 2009, so the middle number is 0. Hence the product of all three numbers is 0.

[*Alternatively*: Let the three numbers be a, b and c.

We have $$a + b = 1004,$$

$$b + c = 1005$$

and $$a + b + c = 2009.$$

Adding the first two equations gives

$$a + 2b + c = 2009$$

and subtracting the third equation from this gives

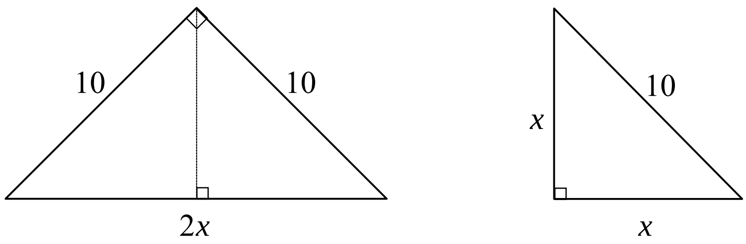
$$b = 0.$$

Thus the product $abc = 0$.]

A5 **$18\frac{1}{3}$** The volume of petrol that Andrea put in, as a fraction of the volume of the tank, is the difference between $\frac{2}{3}$ and $\frac{1}{4}$, which is $\frac{5}{12}$. So she put in $\frac{5}{12}$ of 44 litres and $\frac{5}{12} \times 44 = \frac{5 \times 44}{12} = \frac{5 \times 11}{3} = \frac{55}{3} = 18\frac{1}{3}$.

A6 **10 cm** Since the original triangle is isosceles and right-angled, folding it produces a smaller triangle, also isosceles and right-angled. By Pythagoras' Theorem, the hypotenuse of the original triangle is $\sqrt{200} = 10\sqrt{2}$ cm. Hence the difference between the perimeters of the two triangles is $(10 + 10 + 10\sqrt{2}) - (5\sqrt{2} + 5\sqrt{2} + 10) = 10$ cm.

Alternatively: Let the length of the shorter sides of the new triangle be x cm, shown below. Then the perimeter of the original triangle is $(20 + 2x)$ cm and the perimeter of the new triangle is $(10 + 2x)$ cm. Hence the difference between the perimeters of the two triangles is 10 cm.

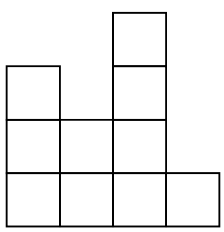


A7 **11** As a fraction of 30 minutes, 30 seconds is $\frac{1}{60}$. So we are considering fractions with a denominator of 60. To obtain a fraction of the required form, the numerator must be a factor of 60 (and less than 60). The numerator can therefore be 1, 2, 3, 4, 5, 6, 10, 12, 15, 20 or 30.

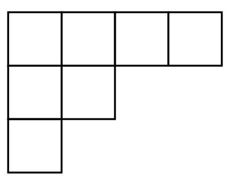
A8 **22π cm** The length of a semicircular arc of radius r is πr and so the total perimeter is $(2 \times (1 + 2 + 4) + 8)\pi = 22\pi$ cm.

A9 **133** After every two numbers, one is omitted. Because $89 = 2 \times 44 + 1$, there must be 44 page numbers missing and so the number on the last page is $89 + 44 = 133$.

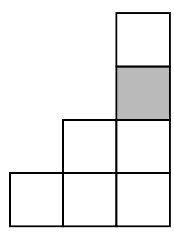
A10 **5000 cm^2** Views from the front and back, the top and bottom, and the two sides are as shown below:



2 lots of 10 faces



2 lots of 7 faces



2 lots of 8 faces
(one hidden from each side)

Each square face has a surface area of 100 cm^2. Hence the total surface area of Gill's shape is $(20 + 14 + 16) \times 100$ cm^2 $= 5000$ cm^2.

B1 In 2007 Alphonse grew twice the number of grapes that Pierre did. In 2008 Pierre grew twice the number of grapes that Alphonse did. Over the two years Alphonse grew 49 000 grapes, which was 7600 less than Pierre. How many grapes did Alphonse grow in 2007?

Solution

Suppose Pierre grew p grapes in 2007. Then, in 2007, Alphonse grew $2p$ grapes.
Thus, in 2008, Alphonse grew $49\,000 - 2p$ and so Pierre grew $98\,000 - 4p$.
Over the two years, the number of grapes Pierre grew was

$$p + (98\,000 - 4p) = 49\,000 + 7600$$

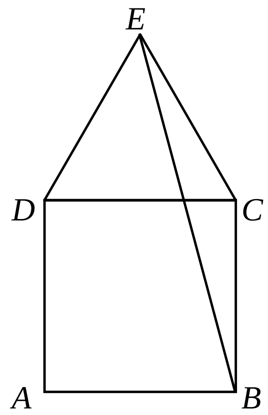
so $$41\,400 = 3p$$

and $$p = 13\,800.$$

Hence, in 2007, Alphonse grew $2 \times 13\,800 = 27\,600$ grapes.

B2 *ABCD* is a square. The point *E* is outside the square so that *CDE* is an equilateral triangle. Find angle *BED*.

Solution



Since *ABCD* is a square, $\angle BCD = 90°$; and since *CDE* is an equilateral triangle, $\angle DCE = 60°$.
Thus $\angle BCE = \angle BCD + \angle DCE = 90° + 60° = 150°$.

Because *CDE* is an equilateral triangle, $EC = DC$ and also, because *ABCD* is a square, $DC = CB$. Hence $EC = CB$ and *ECB* is an isosceles triangle.

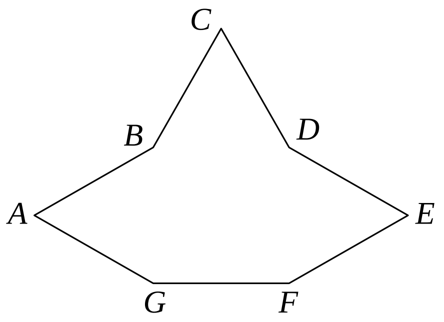
So $\angle CEB = \angle CBE = \frac{1}{2}(180 - 150)° = 15°$, and hence $\angle BED = \angle CED - \angle CEB = 60° - 15° = 45°$.

B3 Tom left a motorway service station and travelled towards Glasgow at a steady speed of 60 mph. Tim left the same service station 10 minutes after Tom and travelled in the same direction at a steady speed, overtaking Tom after a further 1 hour 40 minutes. At what speed did Tim travel?

Solution

Tom travels for 10 minutes longer than Tim, a time of 1 hour and 50 minutes.
Travelling at a speed of 60 mph (or 1 mile per minute), Tom travels a distance of 110 miles.
Tim travelled the same distance in 1 hour and 40 minutes ($1\frac{2}{3}$ hours),
so his speed, in mph, was $110 \div 1\frac{2}{3} = 110 \times \frac{3}{5} = 22 \times 3 = 66$ mph.

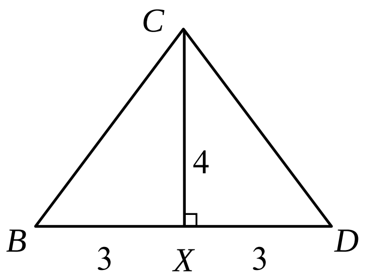
B4 The diagram shows a polygon $ABCDEFG$, in which $FG = 6$ and $GA = AB = BC = CD = DE = EF$. Also $BDFG$ is a square. The area of the whole polygon is exactly twice the area of $BDFG$. Find the length of the perimeter of the polygon.



Solution

The area of square $BDFG$ is $6 \times 6 = 36$ square units.
So the total area of the three triangles ABG, BCD and DEF is also 36 square units.
These three triangles are congruent and so each has an area of 12 square units.

The area of each triangle is $\frac{1}{2} \times \text{base} \times \text{height}$ and the base is 6 units and hence we have $\frac{1}{2} \times 6 \times \text{height} = 12$, so the height is 4 units.

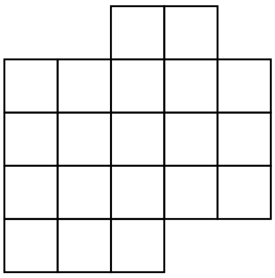


Let X be the midpoint of BD Then CX is perpendicular to the base BD (since BCD is an isosceles triangle).

By Pythagoras' Theorem, $BC = \sqrt{3^2 + 4^2} = 5$ units.

Therefore the perimeter of $ABCDEFG$ is $6 \times 5 + 6 = 36$ units.

B5 An ant wishes to make a circuit of the board shown, visiting each square exactly once and returning to the starting square. At each step the ant moves to an adjacent square across an edge. Two circuits are considered to be the same if the first follows the same path as the second but either starts at a different square or follows the same path in reverse. How many such circuits are possible?



Solution

From Figure 1, we see that there is only one possible route through each of the corner squares, shown shaded, so the ant's path through these squares is as indicated.
There is then only one possible route through square X, giving Figure 2.

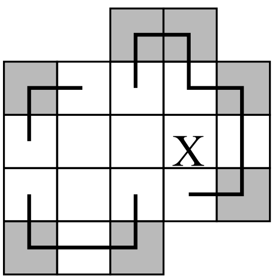


Figure 1

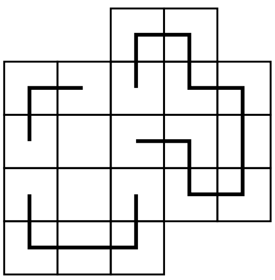


Figure 2

Now consider the shaded square in Figure 3.

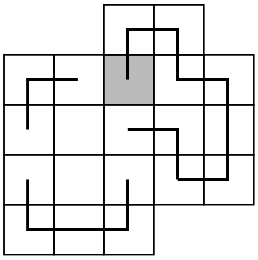


Figure 3

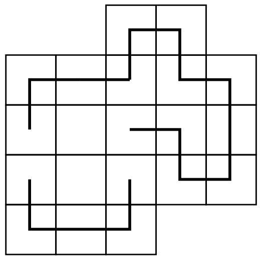


Figure 4

If the path joined this square to the square below, then a closed loop would be formed and the ant could not complete a circuit of the board. Hence the path joins the shaded square to the square on its left (Figure 4).

There are now only two squares which the ant's path has not visited. If a path through all the squares did not join these two, then two loops would be formed instead of a single circuit. We deduce that the path joins these two squares and then there are only two ways of completing the path, as shown in Figure 5.

Figure 5

B6 I want to choose a list of n different numbers from the first 20 positive integers so that no two of my numbers differ by 5. What is the largest value of n for which this is possible? How many different lists are there with this many numbers?

Solution

Any such list contains at most two numbers from the set {1, 6, 11, 16}, at most two numbers from the set {2, 7, 12, 17}, and likewise from each of the sets {3, 8, 13, 18}, {4, 9, 14, 19} and {5, 10, 15, 20}. Hence there are at most $5 \times 2 = 10$ numbers altogether. The list of ten numbers, 1, 2, 3, 4, 5, 11, 12, 13, 14, 15 shows that a selection is indeed possible.

From each of these sets of four numbers of the form $\{a, a + 5, a + 10, a + 15\}$, there are three pairs which do not differ by 5, namely $(a, a + 10\}$, $\{a, a + 15\}$ and $\{a + 5, a + 15\}$. Since we are choosing a pair from each of five such sets, there will be $3^5 = 243$ different lists.