Solutions to the Olympiad Hamilton Paper

H1. Consider five-digit integers that have the following properties. Each of the digits is 1, 2 or 3, and each of 1, 2, 3 occurs at least once as a digit; also, the number is not divisible by 2 nor divisible by 3.

What is the difference between the largest and the smallest of these integers?

Solution

There are four conditions on the five-digit integer:

A each digit is 1, 2 or 3;

B there is at least one occurrence of each of 1, 2 and 3;

C it is not divisible by 2, and so the final digit is either 1 or 3;

D it is not divisible by 3, and so the sum of its digits is not divisible by 3.

The largest and smallest five-digit integers that satisfy both condition A and condition B are 33 321 and 11 123, respectively. Each of these numbers satisfies condition C.

Now $1 + 1 + 1 + 2 + 3 = 8$, which is not divisible by 3, so that 11 123 also satisfies condition D. Therefore 11 123 is the smallest five-digit integer of the required form.

However, $3 + 3 + 3 + 2 + 1 = 12$, which *is* divisible by 3, and hence 33 321 does *not* satisfy condition D. We deduce that, in order to satisfy condition D, a smaller number is required, whose digit sum is not a multiple of 3. The largest such integer less than 33 321 and satisfying all of conditions A, B and C is 33 221, with digit sum $3 + 3 + 2 + 2 + 1 = 11$. It follows that 33 221 is the largest five-digit integer of the required form.

Hence the required answer is 33 221 − 11 123, which equals 22 098.

Alternative

Each integer under consideration is not divisible by 3, and therefore its digit sum is not divisible by 3. But the only digits are 1, 2 and 3, so the number of digits 1 is different from the number of digits 2.

In the smallest such number the digits will be arranged in increasing order from left to right (as far as possible); in the largest such number the digits will be be arranged in decreasing order (as far as possible). Since all three digits 1, 2 and 3 occur, the smallest such number is therefore 11 123 and the largest is 33 221.

We observe that neither of these integers is divisible by 2, so they actually have all four desired properties.

Therefore the difference between the largest and smallest integers with the required properties is 33 221 − 11 123, which equals 22 098.

H2. A rectangle has area 20 cm^2. Reducing the ‘length’ by $2\frac{1}{2}$ cm and increasing the ‘width’ by 3 cm changes the rectangle into a square.

What is the side length of the square?

Solution

Let the length of each side of the square be s cm.

Then the rectangle has length $\left(s + \frac{5}{2}\right)$ cm and width $(s - 3)$ cm. From the information about the area of the rectangle, we therefore have

$$\left(s + \tfrac{5}{2}\right)(s - 3) = 20,$$

which we may expand to obtain

$$s^2 - \frac{s}{2} - \frac{15}{2} = 20,$$

or, on multiplying by 2 and subtracting 40 from both sides,

$$2s^2 - s - 55 = 0.$$

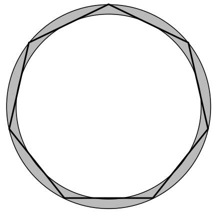
Factorising the left-hand side, we obtain

$$(2s - 11)(s + 5) = 0,$$

from which it follows that $s = \frac{11}{2}$ or $s = -5$. Since negative s has no meaning here, we conclude that the length of each side of the square is $5\frac{1}{2}$ cm.

H3. A regular heptagon is sandwiched between two circles, as shown, so that the sides of the heptagon are tangents of the smaller circle, and the vertices of the heptagon lie on the larger circle. The sides of the heptagon have length 2.

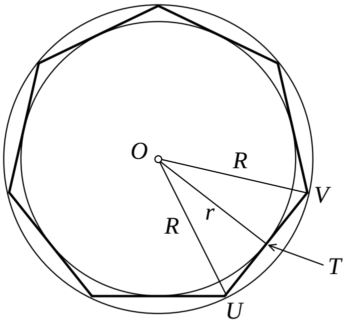
Prove that the shaded *annulus*—the region bounded by the two circles—has area π.



Solution

Let the radius of the larger circle be R and the radius of the smaller circle be r, so that the area of the shaded annulus is $\pi R^2 - \pi r^2$.

Since the heptagon is regular, the two circles have the same centre. The figure shows the common centre O of the two circles, a point of contact T of a side of the heptagon with the smaller circle, and the two vertices U and V of the heptagon adjacent to T. Then $OU = OV = R$ and $OT = r$.



Now $\angle UTO = 90°$ because UV is a tangent and OT is the radius to the point of contact. Thus OT is perpendicular to the base UV of the isosceles triangle OUV, and therefore T is the midpoint of UV. But $UV = 2$, so that $UT = 1$.

By Pythagoras' theorem in triangle OUT, we have $R^2 = r^2 + 1^2 = r^2 + 1$. Hence

$$\begin{aligned}\pi R^2 - \pi r^2 &= \pi\left(R^2 - r^2\right)\\ &= \pi\left(r^2 + 1 - r^2\right)\\ &= \pi,\end{aligned}$$

so that the area of the shaded annulus is π.

Note: There is nothing special about heptagons; the result is true for any regular polygon.

H4. On Monday in the village of Newton, the postman delivered either one, two, three or four letters to each house. The number of houses receiving four letters was seven times the number receiving one letter, and the number receiving two letters was five times the number receiving one letter.

What was the mean number of letters that each house received?

Solution

Let the number of houses receiving one letter on Monday be m, and let the number receiving three letters be n.

Hence, the number of houses receiving four letters was $7m$ and the number of houses receiving two letters on Monday was $5m$.

Thus, the total number of letters delivered was

$$m \times 1 + 7m \times 4 + 5m \times 2 + n \times 3 = 39m + 3n.$$

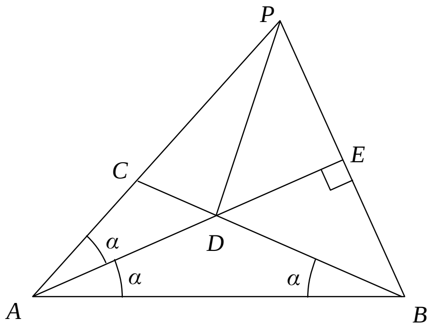
These letters were delivered to $7m + 5m + m + n = 13m + n$ houses, so the mean number of letters that each house received was

$$\frac{39m + 3n}{13m + n} = \frac{3(13m + n)}{13m + n} = 3.$$

H5. Two of the angles of triangle ABC are given by $\angle CAB = 2\alpha$ and $\angle ABC = \alpha$, where $\alpha < 45°$. The bisector of angle CAB meets BC at D. The point E lies on the bisector, but outside the triangle, so that $\angle BEA = 90°$. When produced, AC and BE meet at P.

Prove that $\angle BDP = 4\alpha$.

Solution



We are given that $\angle CAB = 2\alpha$ and the bisector of angle CAB meets BC at D, that is, $\angle CAD = \angle BAD = \alpha$.

An external angle of a triangle equals the sum of the two interior opposite angles, so that $\angle BDE = \angle DAB + \angle DBA = \alpha + \alpha = 2\alpha$.

In the triangles ABE and APE:

(i) $\angle BAE = \angle BAD = \alpha = \angle CAD = \angle PAE$;

(ii) $\angle BEA = 90° = \angle PEA$ (since the sum of the angles on a straight line equals 180°);

(iii) AE is common.

Thus triangles ABE and APE are congruent (AAS). Hence, $EP = EB$.

Now in triangles PED and BED:

(i) $PE = BE$;

(ii) $\angle PED = 90° = \angle BED$;

(iii) ED is common.

Thus triangles PED and BED are congruent (SAS). Hence $\angle PDE = \angle BDE$.

Therefore $\angle BDP = \angle BDE + \angle PDE = 2 \times \angle BDE = 2 \times 2\alpha = 4\alpha$.

H6. Anna and Daniel play a game. Starting with Anna, they take turns choosing a positive integer less than 31 that is not equal to any of the numbers already chosen. The loser is the first person to choose a number that shares a factor greater than 1 with any of the previously chosen numbers.

Does either player have a winning strategy?

Solution

Anna has a winning strategy: she chooses 30 on her first turn. Now $30 = 2 \times 3 \times 5$, so that in order not to lose, neither player can ever select a number not in the list

$$1, 7, 11, 13, 17, 19, 23 \text{ and } 29,$$

since all other positive integers less than 31 are multiples of 2, 3 or 5.

Because no two numbers in this list share a factor greater than 1, whichever number Daniel selects on his turn, Anna may select another one on her next turn, then Daniel may select another one, and so on.

Since eight is an even number, this process can continue for four turns in all by each player, after which all of the numbers will have been selected. At that point Daniel will be forced to select one of the other numbers, which all share a factor greater than 1 with 30, and hence Daniel will lose.

Therefore Anna has a winning strategy.