The United Kingdom Mathematics Trust

Intermediate Mathematical Olympiad and Kangaroo (IMOK)

Olympiad Hamilton Paper

Thursday 20th March 2014

All candidates must be in *School Year 10* (England and Wales), *S3* (Scotland), or *School Year 11* (Northern Ireland).

READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING

1. Time allowed: 2 hours.
2. **The use of calculators, protractors and squared paper is forbidden.**
   Rulers and compasses may be used.
3. Solutions must be written neatly on A4 paper. Sheets must be STAPLED together in the top left corner with the Cover Sheet on top.
4. Start each question on a fresh A4 sheet.
   You may wish to work in rough first, then set out your final solution with clear explanations and proofs.

   Do not hand in rough work.
5. Answers must be FULLY SIMPLIFIED, and EXACT. They may contain symbols such as π, fractions, or square roots, if appropriate, but NOT decimal approximations.
6. Give full written solutions, including mathematical reasons as to why your method is correct.
   Just stating an answer, even a correct one, will earn you very few marks; also, incomplete or poorly presented solutions will not receive full marks.
7. **These problems are meant to be challenging!** The earlier questions tend to be easier; the last two questions are the most demanding.
   Do not hurry, but spend time working carefully on one question before attempting another.
   Try to finish whole questions even if you cannot do many: you will have done well if you hand in full solutions to two or more questions.

DO NOT OPEN THE PAPER UNTIL INSTRUCTED BY THE INVIGILATOR TO DO SO!

The United Kingdom Mathematics Trust is a Registered Charity.
Enquiries should be sent to: Maths Challenges Office,
School of Mathematics, University of Leeds, Leeds, LS2 9JT.
(Tel. 0113 343 2339)
http://www.ukmt.org.uk

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1. Consider five-digit integers that have the following properties. Each of the digits is 1, 2 or 3, and each of 1, 2, 3 occurs at least once as a digit; also, the number is not divisible by 2 nor divisible by 3.

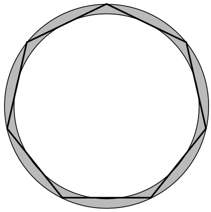
What is the difference between the largest and the smallest of these integers?

2. A rectangle has area 20 cm^2. Reducing the 'length' by $2\frac{1}{2}$ cm and increasing the 'width' by 3 cm changes the rectangle into a square.

What is the side length of the square?

3. A regular heptagon is sandwiched between two circles, as shown, so that the sides of the heptagon are tangents of the smaller circle, and the vertices of the heptagon lie on the larger circle. The sides of the heptagon have length 2.

Prove that the shaded *annulus*—the region bounded by the two circles—has area π.



4. On Monday in the village of Newton, the postman delivered either one, two, three or four letters to each house. The number of houses receiving four letters was seven times the number receiving one letter, and the number receiving two letters was five times the number receiving one letter.

What was the mean number of letters that each house received?

5. Two of the angles of triangle ABC are given by $\angle CAB = 2\alpha$ and $\angle ABC = \alpha$, where $\alpha < 45°$. The bisector of angle CAB meets BC at D. The point E lies on the bisector, but outside the triangle, so that $\angle BEA = 90°$. When produced, AC and BE meet at P.

Prove that $\angle BDP = 4\alpha$.

6. Anna and Daniel play a game. Starting with Anna, they take turns choosing a positive integer less than 31 that is not equal to any of the numbers already chosen. The loser is the first person to choose a number that shares a factor greater than 1 with any of the previously chosen numbers.

Does either player have a winning strategy?