Solutions to the Olympiad Hamilton Paper

1. The digits p, q, r, s and t are all different.
   What is the smallest five-digit integer '$pqrst$' that is divisible by 1, 2, 3, 4 and 5?

   Solution

   Note that all five-digit integers are divisible by 1.

   Next, notice that if a number is divisible by 2 and 5, then it is divisible by 10 and hence has last digit t equal to 0. Since all the digits are different, that means all the other digits are non-zero.

   Among five-digit numbers, those that begin with 1 are smaller than all those which do not. So if we find a number of the required form with first digit 1, then it will be smaller than numbers with larger first digits.

   Similarly, those with first two digits 12 are smaller than all other numbers with distinct non-zero digits. And, in fact, those with first three digits 123 are smaller than all others. Hence if we find such a number with the required properties, it will be smaller than all others.

   So let us try to find a number of the form '123s0'.

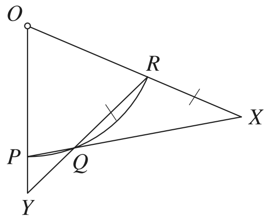
   A number is a multiple of four only if its last two digits form a multiple of four. So we need consider only the case where s is even.

   Similarly, a number is a multiple of three if and only if the sum of its digits is a multiple of three. Since $1 + 2 + 3 + s + 0 = 6 + s$, we only need consider the case where s is a multiple of three.

   Thus 12360 is the only number of the form '123s0' which is divisible by 1, 2, 3, 4 and 5, and as we have explained along the way, it is the smallest number with the required divisibility properties.

2. The diagram shows an arc *PQR* of a circle, centre *O*. The lines *PQ* and *OR* meet at *X*, with $QR = RX$, and the lines *OP* and *RQ* meet at Y.

Prove that $OY = RY$.



Solution

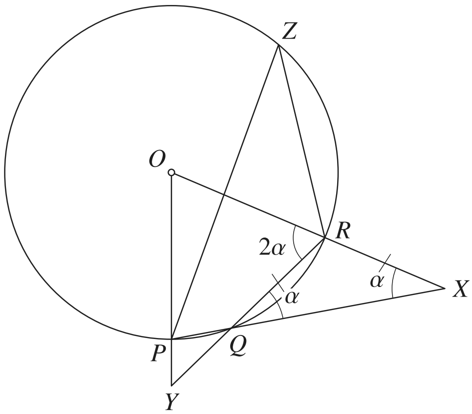
Let $\angle RXQ = \alpha$. Then since $QR = RX$ we have $\angle RQX = \angle RXQ = \alpha$.

Using ‘an external angle of a triangle equals the sum of the two interior opposite angles’ in triangle *RXQ*, we obtain $\angle QRO = \angle RXQ + \angle RQX = \alpha + \alpha = 2\alpha$.

We now proceed in one of two ways: the first method uses circle theorems; the second method uses nothing more than facts about triangles.

Method 1

Let *Z* be any point on the circle which is not on arc *PQR.*



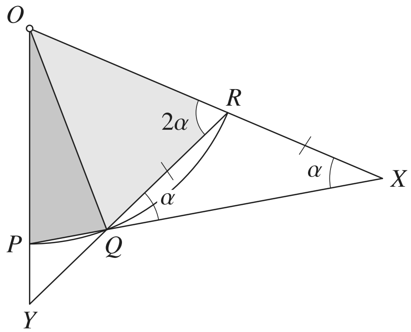
Then $\angle RQX$ is an exterior angle of the cyclic quadrilateral *PQRZ* and therefore $\angle RQX = \angle PZR$. Hence $\angle PZR = \alpha$.

Since ‘the angle at the centre is twice the angle at the circumference’ we deduce that $\angle POR = 2\alpha$.

We therefore have $\angle YOR = \angle POR = 2\alpha = \angle QRO = \angle YRO$, so that $OY = RY$ from ‘sides opposite equal angles’.

Method 2

Since PQR is the arc of a circle, centre O, we have $OP = OQ = OR$ (radii).



In particular, $OQ = OR$, so that triangle OQR is isosceles and $\angle RQO = \angle QRO = 2\alpha$. The sum of the angles on a straight line equals 180°, hence

$$\begin{aligned}\angle OQP &= 180^\circ - \angle RQX - \angle RQO \\ &= 180^\circ - \alpha - 2\alpha \\ &= 180^\circ - 3\alpha.\end{aligned}$$

Now $OP = OQ$, so that triangle OQP is also isosceles and
$\angle OPQ = \angle OQP = 180^\circ - 3\alpha$.

The sum of the angles in a triangle equals 180°, hence, in triangle OPX,

$$\begin{aligned}\angle POX &= 180^\circ - \angle OPX - \angle OXP \\ &= 180^\circ - \angle OPQ - \angle RXQ \\ &= 180^\circ - (180^\circ - 3\alpha) - \alpha \\ &= 2\alpha.\end{aligned}$$

Thus $\angle YOR = \angle POX = 2\alpha = \angle QRO = \angle YRO$, so that $OY = RY$ from 'sides opposite equal angles'.

3. On Monday, the cost of 3 bananas was the same as the total cost of a lemon and an orange.

On Tuesday, the cost of each fruit was reduced by the same amount, resulting in the cost of 2 oranges being the same as the total cost of 3 bananas and a lemon.

On Wednesday, the cost of a lemon halved to 5 p.

What was the cost of an orange on Monday?

Solution

Let x p, y p and z p be the costs on Monday of a banana, lemon and orange respectively. Let the common amount by which the cost of each fruit was reduced on Tuesday be r p.

From the given information, we have

$$3x = y + z, \tag{1}$$

$$2(z - r) = 3(x - r) + y - r, \tag{2}$$

and

$$\tfrac{1}{2}(y - r) = 5. \tag{3}$$

Expanding equation (2), we get $2z - 2r = 3x - 3r + y - r$, so that $2z = 3x - 2r + y$. Substituting for $3x$ from equation (1), we obtain $2z = y + z - 2r + y$, and hence $z = 2y - 2r = 2(y - r)$.

But $y - r = 10$ from equation (3), thus $z = 2 \times 10 = 20$.

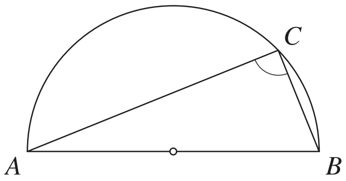
Hence the cost of an orange on Monday was 20 p.

4. The eight points A, B, C, D, E, F, G and H are equally spaced on the perimeter of a circle, so that the arcs AB, BC, CD, DE, EF, FG, GH and HA are all equal.

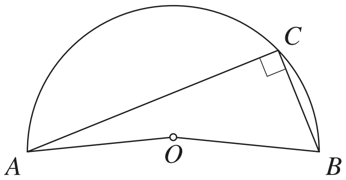
Joining any three of these points forms a triangle. How many of these triangles are right-angled?

Solution

That 'the angle in a semicircle is 90°' is well known: given a diameter AB of a semicircle and another point C on the circumference, as shown, then $\angle BCA = 90°$.



The converse result is also true, but may be less well known: given a right-angled triangle ABC with hypotenuse AB, then AB is a diameter of the circle through A, B and C.



To see why this is true, join A and B to the centre O of the circle. Then, using 'the angle at the centre is twice the angle at the circumference', we obtain $\angle BOA = 2\angle BCA = 2 \times 90° = 180°$. It follows that AOB is a straight line, as required.

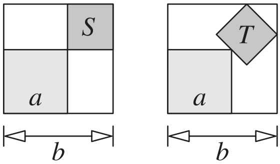
Now consider triangles formed by joining three of the eight points given in the question. The first result shows that the triangle will be right-angled when one side is a diameter, and the second result shows that this is the only way to obtain a right-angled triangle. So we may find the number of right-angled triangles by counting the number of ways of choosing two points at the ends of a diameter, and then choosing the third point.

Now there are 4 ways to choose a diameter connecting the given points: AE, BF, CG and DH.

For a given choice of diameter, there are 6 different ways to choose the third point to form a triangle. For example, diameter AE forms a right-angled triangle with each of the points B, C, D, F, G and H.

Hence altogether there are $4 \times 6 = 24$ ways to form a right-angled triangle.

5. Squares S and T are each placed outside a square of side a and inside a square of side b, as shown. On the left, the *sides* of square S are parallel to the sides of the other two squares; on the right, the *diagonals* of square T are parallel to the sides of the other two squares.

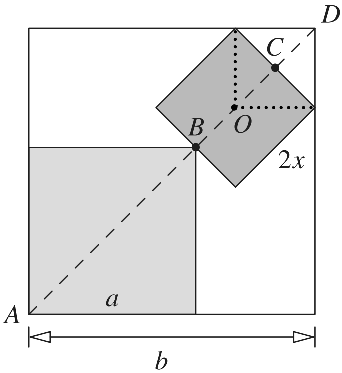


Find the ratio (area of S) : (area of T).

Solution

The area of square S equals $(b - a)^2$.

Let the side length of square T be $2x$. Consider the diagonal AD of the square of side b, shown dashed in the figure.



This diagonal has three parts, the diagonal AB of the square of side a, a line BC crossing the square T, and the height CD of a small triangle. That small triangle is congruent to one quarter of T, as indicated by the dotted lines.

We have $AD = b\sqrt{2}$ and $AB = a\sqrt{2}$ since these are diagonals of squares (using Pythagoras' Theorem). Also $BC = 2x$, and $CD = CO = x$, where O is the centre of the square T.

But $AD = AB + BC + CD$, so we have

$$b\sqrt{2} = a\sqrt{2} + 2x + x$$

from which we deduce that $x = \frac{1}{3}\sqrt{2}\,(b - a)$. Hence the area of T is

$$(2x)^2 = 4x^2 = 4 \times \tfrac{1}{9} \times 2\,(b - a)^2 = \tfrac{8}{9}\,(b - a)^2 .$$

It follows that (area of S) : (area of T) = 9 : 8.

6. Every cell of the following crossnumber is to contain a single digit. All the digits from 1 to 9 are used.

1	2	3
4		
5		

Prove that there is exactly one solution to the crossnumber.

Across	Down
1 A multiple of 21.	1 A multiple of 12.
4 A multiple of 21.	2 A multiple of 12.
5 A multiple of 21.	3 A multiple of 12.

Solution

All multiples of 12 are even, so each digit of 5 ACROSS is 2, 4, 6 or 8. Now 5 ACROSS is a multiple of 21 so it is also a multiple of 3, and we know that a number is a multiple of 3 if, and only if, the sum of the digits is also a multiple of 3. But $2 + 4 + 6 + 8 = 20$, so that the only possibilities are not to use 2, or not to use 8. Hence the digits of 5 ACROSS are 4, 6 and 8, or 2, 4 and 6, in some order. We deduce that 5 ACROSS is 246, 264, 426, 462, 624, 642, 468, 486, 648, 684, 846 or 864. But 5 ACROSS is a multiple of 21 so it is also a multiple of 7. By checking them, we see that only 462 is divisible by 7. Thus 5 ACROSS is 462.

Now $12 = 4 \times 3$, so that all the DOWN answers are multiples of 4, and the number formed by the last two digits of a multiple of 4 is itself divisible by 4. Hence the last two digits of 1 down are 24, 44, 64 or 84. But 2, 4 and 6 have already been placed, therefore the last two digits of 1 DOWN are 84.

The first digit of 4 ACROSS is therefore 8. Now the multiples of 21 between 800 and 900 are 819, 840, 861 and 882. Once again, since 2, 4 and 6 have already been placed 4 ACROSS can only be 819. The last two digits in the DOWN columns are now 84, 16 and 92, all of which are divisible by 4.

Finally, 1 DOWN is a multiple of 3, so it is 384, 684 or 984. But 6 and 9 have already been placed so 1 down is 384. Likewise, 2 DOWN is 516 and 3 down is 792.

Therefore the completed crossnumber can only be

3	5	7
8	1	9
4	6	2

and we may check that all the clues are satisfied. In particular, we observe that we have not used clue 1 ACROSS, so we need to check that 357 is divisible by 21.

Thus there is exactly one solution to the crossnumber.