Solutions to the Olympiad Hamilton Paper

1 If Julie gave £12 to her brother Garron then he would have half the amount that she would have. If instead Garron gave £12 to his sister Julie then she would have three times the amount that he would have.

How much money do they each have?

Solution

Let Julie and Garron have £J and £G respectively. From the information in the question we may form the equations

$$J - 12 = 2(G + 12) \tag{1.1}$$

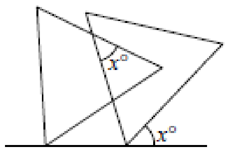
$$J + 12 = 3(G - 12). \tag{1.2}$$

Subtracting equation 1.1 from equation 1.2 gives $24 = G - 60$; thus $G = 84$.

Substituting into equation 1.1 then gives $J - 12 = 2(84 + 12) = 192$. Therefore $J = 204$.

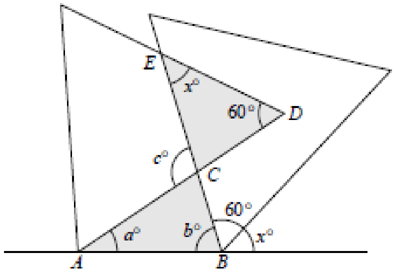
Hence Julie has £204 and Garron has £84.

2. The diagram shows two equilateral triangles. The angles marked $x°$ are equal.
Prove that $x > 30$.



Solution

Each of the angles in an equilateral triangle is equal to 60°. We add two 60° angles to the figure, label three other angles $a°$, $b°$ and $c°$, and label some points, as shown.



At B, since angles on the straight line add up to 180° we have $b + 60 + x = 180$, that is,

$$b = 120 - x. \tag{2.1}$$

Now the angle labelled $c°$ is an exterior angle of triangle CDE, so that $c = x + 60$; it is also an exterior angle of triangle ABC, so that $c = a + b$. Hence $a + b = x + 60$ and therefore, using equation 2.1, we have $a = x + 60 - (120 - x) = 2x - 60$.

But $a > 0$ for the given configuration to occur, hence $2x - 60 > 0$, that is, $x > 30$, as required.

3. A particular four-digit number N is such that:
   (a) the sum of N and 74 is a square; and
   (b) the difference between N and 15 is also a square.

   What is the number N?

Solution

Let

$$N + 74 = x^2 \quad (3.1)$$

$$\text{and} \quad N - 15 = y^2, \quad (3.2)$$

where x and y are different positive integers.

Subtracting equation 3.2 from equation 3.1 gives $89 = x^2 - y^2$. Hence

$$89 = (x - y)(x + y). \quad (3.3)$$

Now x and y are integers, so equation 3.3 gives a factorisation of 89. But 89 is a prime number, so the only possible factors are 1 and 89. Since $x + y > x - y$ we therefore have

$$x + y = 89$$

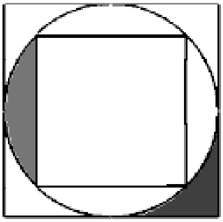
$$x - y = 1.$$

Adding these equations gives $2x = 90$, thus $x = 45$.

Substituting for x in equation 3.1, we obtain $N = 45^2 - 74 = 2025 - 74 = 1951$.

Check: we may also find $y = 44$ and substitute in equation 3.2, to obtain $N = 44^2 + 15 = 1936 + 15 = 1951$.

4. A square just fits within a circle, which itself just fits within another square, as shown in the diagram.
   Find the ratio of the two shaded areas.

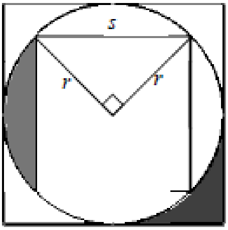


Solution

Let r be the radius of the circle, so that the outer square has side-length $2r$. Thus the total area between the circle and the outer square is

$$(2r)^2 - \pi r^2 = (4 - \pi)r^2.$$

Let the side of the inner square be s. Two radii and one side of this square form a right-angled triangle, as shown.



Applying Pythagoras' theorem to this triangle, we obtain $s^2 = r^2 + r^2 = 2r^2$. Thus the total area between the circle and the inner square is

$$\pi r^2 - s^2 = \pi r^2 - 2r^2 = (\pi - 2)r^2.$$

Since the figure has rotational symmetry of order four, the ratio of the darker shaded area to the lighter shaded area is therefore

$$\frac{(4 - \pi)r^2}{4} : \frac{(\pi - 2)r^2}{4} = (4 - \pi) : (\pi - 2).$$

5. In how many distinct ways can a cubical die be numbered from 1 to 6 so that consecutive numbers are on adjacent faces? Numberings that are obtained from each other by rotation or reflection are considered indistinguishable.

Solution

In any rotation or reflection, adjacent faces remain adjacent, and opposite faces remain opposite. We know that face 1 is adjacent to face 2, and can therefore be opposite to 3, 4, 5 or 6. Without loss of generality we may take 1 as the base. Now consider the four possible top faces in turn.

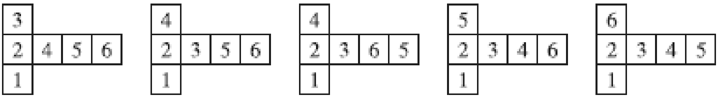
3 at the top Then 2, 4, 5, 6 form the sides. Now 5 has to be adjacent to 4 and 6 is opposite 2, so there is only one such cube possible.

4 at the top Then 2, 3, 5, 6 form the sides. Now 2 and 3 are adjacent, as are 5 and 6, so there are two possible cubes, because we can have 2, 3, 5, 6 (with 2 opposite 5) or 2, 3, 6, 5 (with 2 opposite 6) in order round the cube.

5 at the top Then 2, 3, 4, 6 form the sides. Since 3 is adjacent to 2 and 4, and hence opposite 6, there is just one possibility.

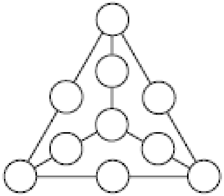
6 at the top Then 2, 3, 4, 5 form the sides. We must have 2 opposite 4, and 3 opposite 5, so there is just one possibility.

Thus there are five distinct ways that a cubical die can be numbered from 1 to 6 so that consecutive numbers are on adjacent faces. The corresponding nets are:



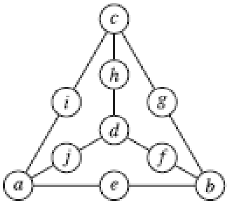
6. Sam wishes to place all the numbers from 1 to 10 in the circles, one to each circle, so that each line of three circles has the same total.

   Prove that Sam's task is impossible.



Solution

Let T be the common total and let the numbers in the circles be a to j, as shown in the figure. Note that a, b, c, d are the numbers which occur in three lines.



Finding the sum of the six lines of three numbers, we obtain

$$3(a + b + c + d) + (e + f + g + h + i + j) = 6T. \qquad (6.1)$$

Now the sum of all the numbers from 1 to 10 equals 55, so that

$$(a + b + c + d) + (e + f + g + h + i + j) = 55.$$

Hence equation 6.1 may be rewritten

$$2(a + b + c + d) + 55 = 6T. \qquad (6.2)$$

But 55 is odd and the other two terms in equation 6.2 are even, which is not possible. We deduce that Sam's task is impossible.