Solutions to the Olympiad Hamilton Paper

1 The sum of three positive integers is 11 and the sum of the cubes of these numbers is 251.

Find all such triples of numbers.

Solution

Let us calculate the first few cubes in order to see what the possibilities are:

$$1^3 = 1,\ 2^3 = 8,\ 3^3 = 27,\ 4^3 = 64,\ 5^3 = 125,\ 6^3 = 216 \text{ and } 7^3 = 343. \quad (*)$$

The sum of the cubes of the positive integers is 251, which is less than 343, hence none of the integers is greater than 6.

Now $\frac{251}{3} = 83\frac{2}{3} > 64 = 4^3$, therefore at least one of the integers is 5 or more.

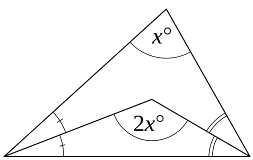
If one of the integers is 6, then the other two cubes add up to $251 - 6^3 = 251 - 216 = 35$. From (*) above, $3^3 + 2^3 = 27 + 8 = 35$ is the only possibility. Also, $6 + 3 + 2 = 11$ so that 6, 3 and 2 is a possible triple of numbers.

If one of the integers is 5, then the other two cubes add up to $251 - 5^3 = 251 - 125 = 126$. From (*) above $5^3 + 1^3 = 125 + 1 = 126$ is the only possibility. Also, $5 + 5 + 1 = 11$ so that 5, 5 and 1 is a possible triple of numbers.

Hence 2, 3, 6 and 1, 5, 5 are the triples of numbers satisfying the given conditions.

2 The diagram shows a triangle and two of its angle bisectors.

What is the value of x?



Solution

Let the sum of the two unlabelled angles in the smaller triangle be y. Then the sum of the two unlabelled angles in the whole triangle is equal to $2y$.

The sum of the angles in a triangle is 180°, hence in the small triangle

$$2x + y = 180 \quad (2.1)$$

and in the whole triangle

$$x + 2y = 180. \quad (2.2)$$

Doubling equation (2.1) and subtracting equation (2.2), we get $3x = 180$ and thus $x = 60$.

3 The first and second terms of a sequence are added to make the third term. Adjacent odd-numbered terms are added to make the next even-numbered term, for example,

first term + third term = fourth term

and third term + fifth term = sixth term.

Likewise, adjacent even-numbered terms are added to make the next odd-numbered term, for example,

second term + fourth term = fifth term.

Given that the seventh term equals the eighth term, what is the value of the sixth term?

Solution

Let a be the first term of the sequence and b the second term. Thus the first eight terms of the sequence are:

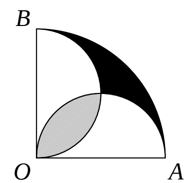
$$a,\ b,\ a+b,\ 2a+b,\ 2a+2b,\ 3a+3b,\ 5a+4b,\ 7a+6b.$$

The seventh term equals the eighth term, hence $5a + 4b = 7a + 6b$. Therefore $2a + 2b = 0$ and so $a = -b$.

Hence the value of the sixth term is $3a + 3b = -3b + 3b = 0$.

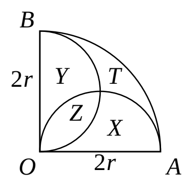
4 The diagram shows a quarter-circle with centre O and two semicircular arcs with diameters OA and OB.

Calculate the ratio of the area of the region shaded grey to the area of the region shaded black.



Solution

Let $2r$ be the radius of the quarter-circle. Hence the radius of each semicircle is r. The diagram is divided into four regions; let their areas be X, Y, Z and T, as shown below.



The area of the quarter-circle is $\frac{1}{4}\pi(2r)^2 = \pi r^2$. The area of each semicircle is $\frac{1}{2}\pi r^2$. Hence $X + Z = \frac{1}{2}\pi r^2$.

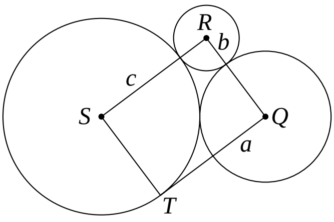
However, the area inside the quarter-circle but outside one semicircle is $\pi r^2 - \frac{1}{2}\pi r^2 = \frac{1}{2}\pi r^2$. This means that $X + T = \frac{1}{2}\pi r^2$.

Therefore $X + T = X + Z$. We conclude that $T = Z$, so that the areas of the shaded regions are equal.

Thus the ratio of the area of the region shaded grey to the area of the region shaded black is $1 : 1$.

5 The diagram shows three touching circles, whose radii are a, b and c, and whose centres are at the vertices Q, R and S of a rectangle $QRST$. The fourth vertex T of the rectangle lies on the circle with centre S.

Find the ratio $a : b : c$.



Solution

In the rectangle $QRST$, we have $QR = TS$ and hence

$$a + b = c. \tag{5.1}$$

In the right-angled triangle QRS, by Pythagoras' Theorem, $QS^2 = QR^2 + RS^2$. But $QS = a + c$, $QR = a + b$ and $RS = b + c$, therefore

$$(a + c)^2 = (a + b)^2 + (b + c)^2 . \tag{5.2}$$

Substituting for a from equation (5.1) into equation (5.2), we get

$$(2c - b)^2 = c^2 + (b + c)^2 .$$

Thus

$$4c^2 - 4bc + b^2 = c^2 + b^2 + 2bc + c^2,$$

so that

$$2c^2 - 6bc = 0$$

and hence

$$c(c - 3b) = 0.$$

But $c \neq 0$, hence $c = 3b$. Again from equation (5.1), $a + b = 3b$ and thus $a = 2b$.

Therefore the ratio $a : b : c = 2 : 1 : 3$.

6 In the diagram, the number in each cell shows the number of shaded cells with which it shares an edge or a corner. The total of all the numbers for this shading pattern is 16. Any shading pattern obtained by rotating or reflecting this one also has a total of 16.

2	1	2
3	2	2
1	2	1

Prove that there are exactly two shading patterns (not counting rotations or reflections) which have a total of 17.

Solution

Whenever a cell is shaded, one is added to all the cells with which it shares an edge or corner. So consider an alternative numbering system: in each shaded cell write the number of cells with which it shares an edge or corner; leave each unshaded cell blank. For example, for the shading pattern given in the question we obtain:

	5	
	8	
3		

This is equivalent to the original numbering system; in particular, the total of all the numbers is the same.

Now a shaded corner cell has 3 adjacent cells; a shaded edge cell has 5 adjacent cells; the shaded central cell has 8 adjacent cells. Thus the total of all the numbers for a shading pattern is made up solely by adding multiples of 3, 5 and 8.

For a 3×3 diagram the available numbers are therefore: four 3s, four 5s and one 8.

If the 8 is used, a remaining total of $17 - 8 = 9$ is required. The only way to attain 9 is to use three 3s.

If the 8 is not used, since 17 is not a multiple of 3 at least one 5 is needed. Now $17 - 1 \times 5 = 12$, $17 - 2 \times 5 = 7$ and $17 - 3 \times 5 = 2$, but neither 7 nor 2 is a multiple of 3. So the only possibility is to use one 5 and then a remaining total of 12 is required. The only way to attain 12 is to use four 3s.

Thus the only possibilities are: 3, 3, 3, 3, 5 and 3, 3, 3, 8. Both of these are possible using the available numbers. What are the corresponding shading patterns?

3	5	3
3		3

3		3
	8	
3		

The diagrams above give examples of the only possible shading pattern for each set of numbers—all others are rotations of one of these. In the first case, the four corners are shaded to obtain four 3s, then there is only one way, up to rotation, to shade an edge cell to obtain the 5. In the second case, the centre is shaded to obtain the 8, then there is only one way, up to rotation, to shade three corner cells to obtain three 3s.

Therefore there are exactly two shading patterns with a total of 17.