Solutions to the Olympiad Hamilton Paper

1. An aquarium contains 280 tropical fish of various kinds. If 60 more clownfish were added to the aquarium, the proportion of clownfish would be doubled.

   How many clownfish are in the aquarium?

   Solution
   Let there be x clownfish in the aquarium.
   If 60 clownfish are added there are $x + 60$ clownfish and 340 tropical fish in total.
   Since the proportion of clownfish is then doubled, we have

$$2 \times \frac{x}{280} = \frac{x + 60}{340}.$$

   Multiplying both sides by 20, we get

$$\frac{x}{7} = \frac{x + 60}{17}$$

   and hence

$$17x = 7(x + 60).$$

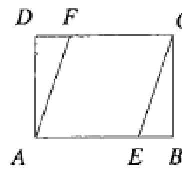
   It follows that $x = 42$ and thus there are 42 clownfish in the aquarium.

2. Find the possible values of the digits p and q, given that the five-digit number '$p543q$' is a multiple of 36.

   Solution
   Since '$p543q$' is a multiple of 36 it is a multiple of both 9 and 4.
   The sum of the digits of a multiple of 9 is also a multiple of 9, hence $p + 5 + 4 + 3 + q$ is a multiple of 9. But $5 + 4 + 3 = 12$ and each of p and q is a single digit, so that $p + q = 6$ and $p + q = 15$ are the only possibilities.
   Since '$p543q$' is a multiple of 4 and '$p5400$' is always divisible by 4, it follows that '$3q$' is divisible by 4. The only possible values for '$3q$' are 32 and 36, so that $q = 2$ or $q = 6$.
   If $q = 2$, then $p + q = 15$ is not possible since p is a single digit. Hence $p + q = 6$ and so $p = 4$.
   If $q = 6$, then $p + q = 6$ is not possible since '$p543q$' is a five-digit number and therefore the digit p cannot be zero. Hence $p + q = 15$ and so $p = 9$.
   Therefore $p = 4$, $q = 2$ and $p = 9$, $q = 6$ are the only possible values of the digits p and q.

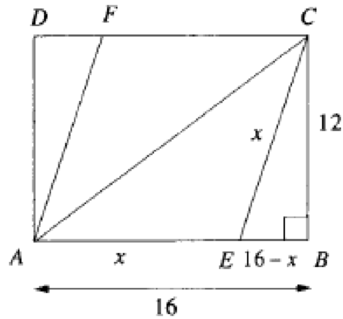
3. In the diagram, $ABCD$ is a rectangle with $AB = 16$ cm and $BC = 12$ cm. Points E and F lie on sides AB and CD so that $AECF$ is a rhombus.

What is the length of EF?



Solution

Let the sides of the rhombus $AECF$ have length x cm. Hence $AE = x$ and $EB = 16 - x$. Since $ABCD$ is a rectangle, angle EBC is a right angle.



Using Pythagoras' theorem in triangle ABC, we have $AC^2 = 16^2 + 12^2 = 400$, so that $AC = 20$ cm.

Using Pythagoras' theorem in triangle EBC, we have

$$EC^2 = CB^2 + EB^2$$

and hence

$$x^2 = 12^2 + (16 - x)^2,$$

which can be rearranged to give

$$x^2 = 144 + 256 - 32x + x^2.$$

It follows that

$$32x = 400$$

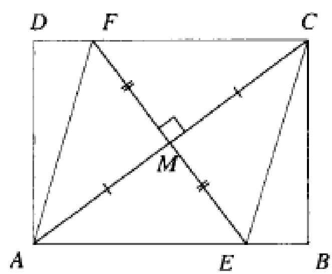
and so

$$x = \frac{25}{2}.$$

We may now proceed in various ways; we show two different methods.

First method

Let M be the point of intersection of the diagonals AC and EF of $AECF$. Since $AECF$ is a rhombus, angle FMC is a right angle and M is the mid-point of both AC and EF.



Using Pythagoras' theorem in triangle FMC, we have

$$CF^2 = (\tfrac{1}{2}AC)^2 + (\tfrac{1}{2}EF)^2$$

and hence

$$\left(\frac{25}{2}\right)^2 = 10^2 + (\tfrac{1}{2}EF)^2.$$

It follows that

$$625 = 400 + EF^2$$

and so

$$EF = 15.$$

Therefore the length of EF is 15 cm.

Second method

We make use of the fact that

area of rhombus $AECF$ = area of rectangle $ABCD$ $-$ 2 $\times$ area of triangle EBC.

Now the area of a rhombus is half the product of its diagonals. Also, the area of triangle EBC is $\tfrac{1}{2}EB \times BC$ and $EB = 16 - \tfrac{25}{2} = \tfrac{7}{2}$. We therefore have

$$\tfrac{1}{2}AC \times EF = 16 \times 12 - \tfrac{7}{2} \times 12.$$

Hence

$$10 \times EF = 192 - 42 = 150$$

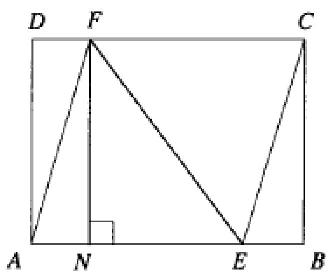
and so

$$EF = 15.$$

Therefore the length of EF is 15 cm.

Remark

Another method uses Pythagoras' theorem in the right-angled triangle ENF shown below.



4. Four positive integers a, b, c and d are such that:

   the sum of a and b is half the sum of c and d;

   the sum of a and c is twice the sum of b and d;

   the sum of a and d is one and a half times the sum of b and c.

   What is the smallest possible value of $a + b + c + d$?

Solution

There are three equations here but four unknown values, a, b, c and d. Thus it is not possible just to solve the equations to find the values of a, b, c and d. What we can do is to find relationships between them and then deduce possible values of a, b, c and d.

From the given information,

$$a + b = \tfrac{1}{2}(c + d) \tag{1}$$

$$a + c = 2(b + d) \tag{2}$$

$$a + d = \tfrac{3}{2}(b + c). \tag{3}$$

We may proceed in various ways; we show two methods, substitution and elimination.

First method: substitution

From (1), we have

$$a = -b + \tfrac{1}{2}(c + d). \tag{4}$$

Substituting in (2), we get

$$-b + \tfrac{1}{2}(c + d) + c = 2(b + d)$$

and hence

$$\tfrac{3}{2}c - \tfrac{3}{2}d = 3b,$$

that is,

$$c - d = 2b. \tag{5}$$

Substituting from (4) in (3), we get

$$-b + \tfrac{1}{2}(c + d) + d = \tfrac{3}{2}(b + c)$$

so that

$$-c + \tfrac{3}{2}d = \tfrac{5}{2}b. \tag{6}$$

Now adding (5) and (6) we obtain

$$\tfrac{1}{2}d = \tfrac{9}{2}b,$$

and hence

$$d = 9b.$$

Once we have minimised $b + d$, then we automatically minimise $a + c$, because of equation (2), and hence minimise the sum we are interested in.

Since b and d are positive integers, $b = 1$ and $d = 9$ are the smallest possible values with $d = 9b$. From (5) and (4), we see that the corresponding values of c and a are $c = 11$ and $a = 9$, both of which are also positive integers, as required.

Checking these values in equations (1) to (3), we confirm that they are valid solutions of the given equations.

Hence the smallest possible value of $a + b + c + d$ is 30.

Second method: elimination

We may rearrange the three equations (1), (2) and (3) to give

$$2a + 2b = c + d \tag{7}$$

$$a + c = 2b + 2d \tag{8}$$

$$2a + 2d = 3b + 3c. \tag{9}$$

Adding (7) and (8), we get

$$3a + 2b + c = 2b + c + 3d$$

and hence

$$a = d.$$

Then (7) and (9) may be rewritten

$$2b - c + d = 0. \tag{10}$$

and

$$3b + 3c - 4d = 0. \tag{11}$$

Now adding $3 \times$ (10) and (11), we obtain

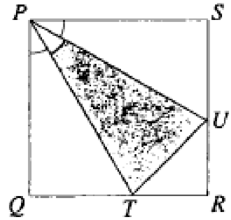
$$9b - d = 0$$

and hence

$$d = 9b.$$

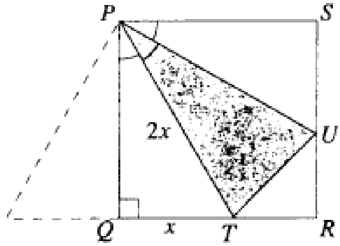
The solution now proceeds in the same way as the first method.

5. The diagram shows a triangle PTU inscribed in a square $PQRS$. Each of the marked angles at P is equal to 30°.
Prove that the area of the triangle PTU is one third of the area of the square $PQRS$.



First solution

Let $QT = x$, so that $PT = 2x$, since triangle PTQ is half an equilateral triangle.



Using Pythagoras' theorem in triangle PTQ, we get

$$\begin{aligned} PQ^2 &= PT^2 - QT^2 \\ &= (2x)^2 - x^2 \\ &= 3x^2 \end{aligned}$$

and hence

$$PQ = \sqrt{3}x.$$

We can now find the areas of the three unshaded right-angled triangles.

$$\begin{aligned} \text{Area of triangle } PQT &= \tfrac{1}{2} \times x \times \sqrt{3}x \\ &= \frac{\sqrt{3}}{2}x^2. \end{aligned}$$

Similarly,

$$\text{area of triangle } PSU = \frac{\sqrt{3}}{2}x^2.$$

Finally,

$$\begin{aligned} \text{area of triangle } TRU &= \tfrac{1}{2} \times \left(\sqrt{3}x - x\right) \times \left(\sqrt{3}x - x\right) \\ &= \tfrac{1}{2}\left(3x^2 - 2\sqrt{3}x^2 + x^2\right) \\ &= 2x^2 - \sqrt{3}x^2. \end{aligned}$$

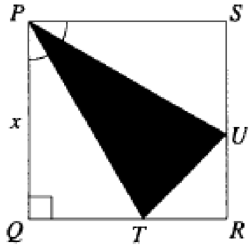
Therefore the total unshaded area is

$$\frac{\sqrt{3}}{2}x^2 + \frac{\sqrt{3}}{2}x^2 + 2x^2 - \sqrt{3}x^2 = 2x^2.$$

However, the area of the square $PQRS$ is $\left(\sqrt{3}x\right)^2 = 3x^2$. It follows that the shaded area is x^2, which is one third of the area of the square.

Second solution

Let the sides of the square $PQRS$ have length x.



Then in triangle PQT we have

$$\cos 30^\circ = \frac{x}{PT},$$

and hence

$$PT = \frac{x}{\cos 30^\circ}.$$

Now by symmetry $PU = PT$ so that

$$\begin{aligned}\text{area of triangle } PTU &= \tfrac{1}{2}PT \times PU \sin \angle TPU \\ &= \tfrac{1}{2}x^2 \frac{\sin 30^\circ}{\cos^2 30^\circ}.\end{aligned}$$

Now $\cos 30^\circ = \frac{\sqrt{3}}{2}$ and $\sin 30^\circ = \frac{1}{2}$. Therefore

$$\begin{aligned}\text{area of triangle } PTU &= \tfrac{1}{2}x^2 \times \frac{\frac{1}{2}}{\frac{3}{4}} \\ &= \tfrac{1}{3}x^2.\end{aligned}$$

Hence the area of the triangle PTU is one third of the area of the square $PQRS$.

6. Two different cuboids are placed together, face-to-face, to form a large cuboid.

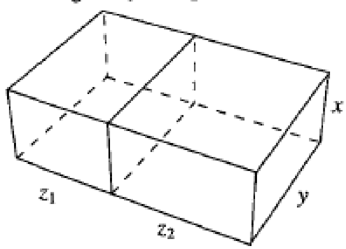
The surface area of the large cuboid is $\frac{3}{4}$ of the total surface area of the original two cuboids.

Prove that the lengths of the edges of the large cuboid may be labelled x, y and z, where

$$\frac{2}{z} = \frac{1}{x} + \frac{1}{y}.$$

First solution

Since the two cuboids are placed together, face-to-face, to form a large cuboid, two of the edges have the same lengths. Let these common lengths be x and y, and let the other edges of the two cuboids have lengths z_1 and z_2, as shown.



Now the surface area of the large cuboid is $\frac{3}{4}$ of the total surface area of the original two cuboids. Therefore

$$2\left[xy + x(z_1 + z_2) + y(z_1 + z_2)\right] = \tfrac{3}{4}\left[2(xy + z_1x + z_1y) + 2(xy + z_2x + z_2y)\right]$$

so that

$$4\left[xy + x(z_1 + z_2) + y(z_1 + z_2)\right] = 3\left[2xy + x(z_1 + z_2) + y(z_1 + z_2)\right]$$

and hence

$$2xy = x(z_1 + z_2) + y(z_1 + z_2),$$

that is,

$$2xy = (x + y)(z_1 + z_2).$$

Now $z_1 + z_2 = z$, where z is the edge length of the large cuboid. Therefore

$$2xy = (x + y)z$$

which may be rearranged to give

$$\frac{2}{z} = \frac{x + y}{xy}$$

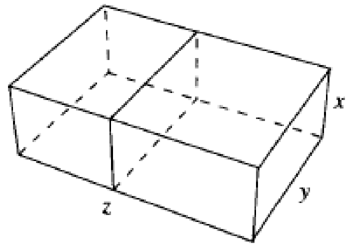
$$= \frac{1}{y} + \frac{1}{x}.$$

Hence the lengths of the edges of the large cuboid may be labelled x, y and z, where

$$\frac{2}{z} = \frac{1}{x} + \frac{1}{y}.$$

Second solution

Let the large cuboid have dimensions x, y and z, as shown.



Now the total surface area T of the two original cuboids is equal to the surface area of the large cuboid added to the area of the two faces which are joined together. But the surface area of the large cuboid is $\frac{3}{4}T$, hence the area of the two faces which are joined together is $\frac{1}{4}T$, that is, $\frac{1}{3}$ of the surface area of the large cuboid.

Therefore

$$2xy = \tfrac{1}{3}(2xy + 2yz + 2zx)$$

so that

$$6xy = 2xy + 2yz + 2zx$$

and hence

$$2xy = yz + zx.$$

Dividing by xyz, we obtain, as required,

$$\frac{2}{z} = \frac{1}{x} + \frac{1}{y}.$$