Solutions to the Olympiad Hamilton Paper

1. How many four-digit multiples of 9 consist of four different odd digits?

First solution

There are five odd digits: 1, 3, 5, 7 and 9.

The sum of the four smallest odd digits is 16 and the sum of the four largest is 24. Hence the digit sum of any four-digit number with different odd digits lies between 16 and 24, inclusive.

However, the sum of the digits of a multiple of 9 is also a multiple of 9, and the only multiple of 9 between 16 and 24 is 18. Hence the sum of the four digits is 18.

Now $1 + 3 + 5 + 9 = 18$, so that the four digits *can* be 1, 3, 5 and 9. If 7 is one of the four digits then the sum of the other three is 11, which is impossible. So 7 cannot be one of the digits and therefore the four digits can only be 1, 3, 5 and 9.

The number of arrangements of these four digits is $4 \times 3 \times 2 \times 1 = 24$. Hence there are 24 four-digit multiples of 9 that consist of four different odd digits.

Second solution

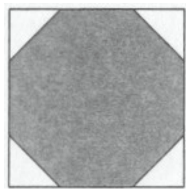
The sum of all five odd digits is $1 + 3 + 5 + 7 + 9 = 25$.

Subtracting 1, 3, 5, 7 and 9 in turn we get 24, 22, 20, 18 and 16, only one of which is a multiple of 9, namely $18 = 25 - 7$. Since the sum of the digits of a multiple of 9 is also a multiple of 9, it follows that the four digits can only be 1, 3, 5 and 9.

The number of arrangements of these four digits is $4 \times 3 \times 2 \times 1 = 24$. Hence there are 24 four-digit multiples of 9 that consist of four different odd digits.

2. A regular octagon with sides of length a is inscribed in a square with sides of length 1, as shown.

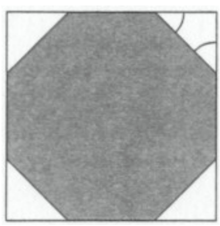
Prove that $a^2 + 2a = 1$.

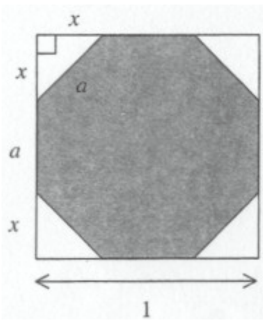


First solution

Consider each of the four unshaded triangles. The angle at the vertex of a square is 90° so each triangle is right-angled.

The marked angles in the left-hand diagram are both external angles of a regular octagon, so each is equal to $\frac{1}{8} \times 360° = 45°$. Hence each triangle is isosceles (since sides opposite equal angles are equal).





Let the two equal sides of one of these triangles have length x, as shown in the right-hand diagram.

From Pythagoras' theorem $\qquad x^2 + x^2 = a^2$

so that $\qquad 2x^2 = a^2$

and hence $\qquad x = \dfrac{a}{\sqrt{2}}.$

Now the side of the square has length 1, therefore

$$a + 2x = 1,$$

that is, $\qquad a + a\sqrt{2} = 1,$

or $\qquad a\sqrt{2} = 1 - a.$

Squaring this equation we get

$$2a^2 = 1 - 2a + a^2$$

and therefore $\qquad a^2 + 2a = 1.$

Second solution

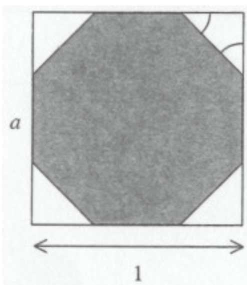
We derive an equation for a using

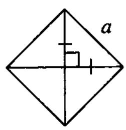
$$\text{unshaded area + area of octagon = area of the square.} \qquad (1)$$

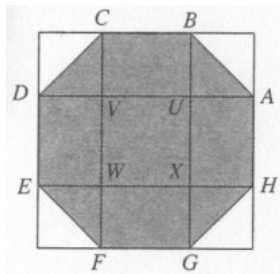
Consider each of the four unshaded triangles. The angle at the vertex of a square is 90° so each triangle is right-angled.

The marked angles in the left-hand diagram are both external angles of a regular octagon, so each is equal to $\frac{1}{8} \times 360° = 45°$. Hence each triangle is isosceles (since sides opposite equal angles are equal).

Therefore each of the four unshaded triangles is isosceles and right-angled, with hypotenuse of length a, so the four triangles can be reassembled to form a square of side a (see below). Hence the unshaded area is equal to a^2.







Similarly, the four shaded triangles in the right-hand figure together have an area of a^2. The octagon comprises these four shaded triangles together with two rectangles, *ADEH* and *BCFG*, which overlap in the square *UVWX*. Therefore the area of the octagon is

$$a^2 + \text{area}\,ADEH + \text{area}\,BCFG - \text{area}\,UVWX.$$

But the two rectangles each have area $a \times 1$ and the area of square *UVWX* is $a \times a$, so that the octagon has area

$$a^2 + a + a - a^2 = 2a.$$

Finally, the large square has area 1, so equation (1) gives

$$a^2 + 2a = 1.$$

3. Kelly cycles to a friend's house at an average speed of 12 km/hr. Her friend is out, so Kelly immediately returns home by the same route. At what average speed does she need to cycle home if her average speed over the whole journey is to be 15 km/hr?

First solution

Let the distance cycled to the house be d km; let the time taken on the journey to the friend's house be t_1 hours and let the time taken on the way back be t_2 hours. From the given information about average speeds,

$$12 = \frac{d}{t_1}$$

and

$$15 = \frac{2d}{t_1 + t_2}.$$

These equations may be rearranged to give

$$12t_1 = d \qquad (1)$$

and

$$15t_1 + 15t_2 = 2d. \qquad (2)$$

Substituting from equation (1) into equation (2), we get

$$15t_1 + 15t_2 = 24t_1$$

so that

$$t_1 = \tfrac{5}{3}t_2.$$

Then equation (1) gives

$$12 \times \tfrac{5}{3}t_2 = d,$$

and hence

$$20 = \frac{d}{t_2}.$$

Thus Kelly's average speed cycling home is 20 km/h.

Second solution

Let the distance cycled to the house be d km; let the average speed for the journey home be v km/h. Then from the information given

$$\text{the time for the outward journey} = \frac{d}{12} \text{ hours,}$$

$$\text{the time for the homeward journey} = \frac{d}{v} \text{ hours,}$$

$$\text{and the time for the whole journey} = \frac{2d}{15} \text{ hours.}$$

Therefore we have

$$\frac{d}{12} + \frac{d}{v} = \frac{2d}{15},$$

which may be rearranged to give

$$\frac{1}{v} = \frac{2}{15} - \frac{1}{12} = \frac{8 - 5}{60} = \frac{1}{20}.$$

Hence $v = 20$ and Kelly's average speed cycling home is 20 km/h.

4. A triangle is bounded by the lines whose equations are $y = -x - 1$, $y = 2x - 1$ and $y = k$, where k is a positive integer.
For what values of k is the area of the triangle less than 2008?

Solution

The lines with equations $y = -x - 1$ and $y = 2x - 1$ intersect when

$$-x - 1 = 2x - 1,$$

from which $$x = 0,$$

so that the lines meet at $(0, -1)$.
The line $y = k$ intersects the line $y = -x - 1$ when

$$k = -x - 1,$$

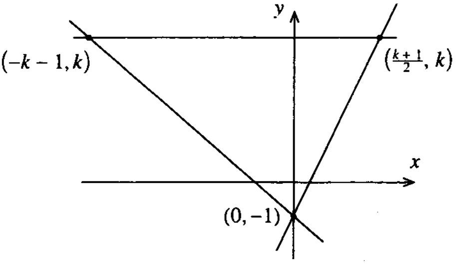
from which $$x = -k - 1,$$

and the line $y = k$ intersects the line $y = 2x - 1$ when

$$k = 2x - 1$$

from which $$x = \frac{k+1}{2}.$$

Thus the three intersection points are $(0, -1)$, $(-k - 1, k)$ and $\left(\frac{k+1}{2}, k\right)$.



Now the enclosed triangle has height $k + 1$ and 'base' equal to

$$\frac{k+1}{2} - (-k - 1) = \frac{k+1}{2} + k + 1$$
$$= \tfrac{3}{2}(k + 1),$$

so the enclosed area is

$$\tfrac{1}{2} \times \tfrac{3}{2}(k + 1) \times (k + 1) = \tfrac{3}{4}(k + 1)^2.$$

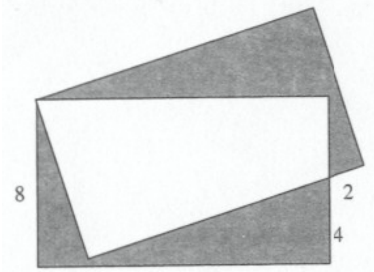
Therefore when the area is less than 2008,

$$\tfrac{3}{4}(k + 1)^2 < 2008,$$

so that $$(k + 1)^2 < \frac{8032}{3}$$
$$= 2677\tfrac{1}{3}.$$

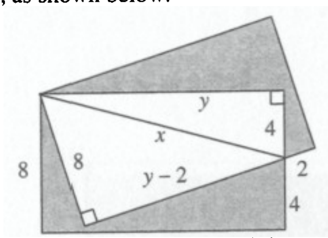
Now $51^2 = 2601$ and $52^2 = 2704$ so that $k + 1 < 52$, that is, $k < 51$. Hence the possible values of k are given by $1 \leqslant k \leqslant 50$.

5. Two congruent rectangles have a common vertex and overlap as shown in the diagram. What is the total shaded area?



Solution

Let the unknown side of the rectangle have length y and let one diagonal of the unshaded quadrilateral have length x, as shown below.



Applying Pythagoras' theorem to the two unshaded right-angled triangles we get

$$x^2 = 8^2 + (y - 2)^2$$

and $$x^2 = y^2 + 4^2.$$

Eliminate x^2 from these equations to give

$$64 + (y - 2)^2 = y^2 + 16,$$

that is, $$64 + y^2 - 4y + 4 = y^2 + 16,$$

which rearranges to $$52 = 4y$$

and hence $$y = 13.$$

Now the shaded area is equal to twice the area of one rectangle minus twice the area of the unshaded region, that is,

$$2(8 \times 13) - 2(\tfrac{1}{2} \times 8 \times 11 + \tfrac{1}{2} \times 4 \times 13) = 68.$$

Hence the total shaded area equals 68.

6. Find all solutions to the simultaneous equations

$$x^2 - y^2 = -5$$

$$2x^2 + xy - y^2 = 5.$$

First solution

We may rewrite the given equations by factorising the left-hand sides:

$$(x - y)(x + y) = -5 \qquad (1)$$

$$(2x - y)(x + y) = 5. \qquad (2)$$

Since −5 is non-zero, we may divide (2) by (1) to get

$$\frac{2x - y}{x - y} = -1,$$

which rearranges to $2x - y = y - x$

and hence $x = \tfrac{2}{3}y.$

Now substitute $x = \tfrac{2}{3}y$ in $x^2 - y^2 = -5$ to obtain

$$\tfrac{4}{9}y^2 - y^2 = -5,$$

so that $y^2 = 9$

and hence $y = \pm 3.$

Since $x = \tfrac{2}{3}y$ we deduce that the equations have two solutions:

$$x = 2, y = 3 \quad \text{and} \quad x = -2, y = -3.$$

Second solution

The given equations are

$$x^2 - y^2 = -5 \tag{3}$$

$$2x^2 + xy - y^2 = 5. \tag{4}$$

Adding (3) and (4) we get

$$3x^2 + xy - 2y^2 = 0$$

which factorises to $(3x - 2y)(x + y) = 0.$

Hence $x = \tfrac{2}{3}y$ or $x = -y$. But, from equation (3), we know $x \neq -y$ so we have $x = \tfrac{2}{3}y$.

Substitute $x = \tfrac{2}{3}y$ in (3) to obtain

$$\tfrac{4}{9}y^2 - y^2 = -5,$$

so that $y^2 = 9$

and hence $y = \pm 3.$

Since $x = \tfrac{2}{3}y$ we deduce that the equations have two solutions:

$$x = 2, y = 3 \quad \text{and} \quad x = -2, y = -3.$$