Solutions to the Olympiad Hamilton Paper

1. Let the number in the bottom left-hand block be n. By rule (a) the other values on the bottom row are $2n$, $4n$ and $8n$. By rule (b) the values in the second row are $3n$, $6n$ and $12n$, those in the third row are $9n$ and $18n$, and the fourth row is $27n$. Hence the sum of all ten numbers is $90n$.

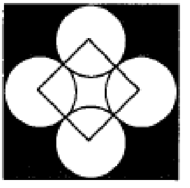
   Since $90 = 2 \times 3^2 \times 5$ the smallest value of n for which $90n$ is a cube is $2^2 \times 3 \times 5^2 = 300$.

2. Let x be the smallest positive integer in the sequence.

   From the information given, $x^2 + (x+1)^2 + (x+2)^2 = (x+3)^2 + (x+4)^2$, which can be rearranged to $x^2 - 8x - 20 = 0$. Hence $(x-10)(x+2) = 0$ so that $x = 10$ or $x = -2$. But x is positive and therefore $x = 10$ is the only solution.

   This proves that there is exactly one such sequence.

3. From the given information, the four circles are symmetrically arranged in the square and therefore joining the centres of the four circles forms a square of side two and area four. The remaining unshaded area is made up of four three-quarter circles of total area $4 \times \frac{3}{4} \times \pi \times 1^2$, which equals 3π. Hence the unshaded area is $4 + 3\pi$.



   By Pythagoras' theorem, the distance between the centres of two non-touching circles is $\sqrt{2^2 + 2^2} = 2\sqrt{2}$. Hence the given square has side $2\sqrt{2} + 2 \times 1$. Therefore

$$\begin{aligned}\text{the shaded area} &= \left(2 + 2\sqrt{2}\right)^2 - (4 + 3\pi)\\ &= 4 + 8\sqrt{2} + 8 - 4 - 3\pi\\ &= 8\left(1 + \sqrt{2}\right) - 3\pi.\end{aligned}$$

4. Let the speed at which Inzamam walks be v miles per minute . Hence Inzamam runs at a speed of $2v$ miles per minute.

   Let the time taken on Monday be t minutes, so that the time taken on Tuesday is $(t - 6)$ minutes.

   Let the distance to school be s miles.

   Hence on Monday $\quad s = \frac{2}{3}t \times v + \frac{1}{3}t \times 2v = \frac{4}{3}tv$

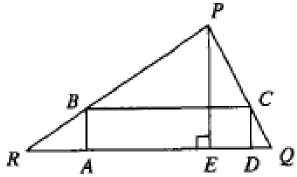
   and on Tuesday $\quad s = \frac{2}{3}(t-6) \times 2v + \frac{1}{3}(t-6) \times v = \frac{5}{3}v(t-6)$.

   Hence $\frac{4}{3}tv = \frac{5}{3}v(t-6)$ which can be rearranged to give $4t = 5(t-6)$ and therefore $t = 30$.

   Thus Inzamam walks for twenty minutes and runs for ten minutes on Monday.

   Since Inzamam runs twice as fast as he walks, the time taken to walk to school on Wednesday is $20 + 2 \times 10 = 40$ minutes.

5. Let the perpendicular from P to RQ meet RQ at E. We are given that the height of the rectangle, AB, is one third of the perpendicular height of triangle RPQ.



Since $\angle BRA = \angle PRE$ and $\angle RAB = \angle REP$, triangles RBA and RPE are similar and
$RA : RE = AB : EP = 1 : 3$. Hence $AE : RE = 2 : 3$.
Likewise, triangles QDC and QEP are similar and
$AE : RE = 2 : 3$.
Hence $(AE + ED) : (RE + EQ) = 2 : 3$, that is,
$ED : EQ = 2 : 3$.

Now the ratio of the area of the rectangle to the area of the triangle is

$$(BA \times AD) : (\tfrac{1}{2}RQ \times PE) = (\tfrac{1}{3}PE \times \tfrac{2}{3}RQ) : (\tfrac{1}{2}RQ \times PE)$$
$$= \tfrac{2}{9} : \tfrac{1}{2}$$
$$= 4 : 9.$$

6. Let C be the sum of the four corner numbers. Now the sum of the numbers 1 to 10 is 55, so that the sum of the numbers in the two rows and two columns is $55 + C$ and therefore $4T = 55 + C$ (*).

(a) When $T = 20$, from (*) we have $4 \times 20 = 55 + C$ so that $C = 25$. One possible solution is

10	2	5	3
6			9
4	1	7	8

(b) The smallest sum of four of the ten numbers is $1 + 2 + 3 + 4 = 10$. Therefore $C \geq 10$ so that $4T \geq 55 + 10$, from (*), and hence $T > 16$.
Suppose $T = 17$. From (*) we have $4 \times 17 = 55 + C$ so that $C = 13$. Let the middle numbers in the two columns be x and y. Then the sum of these columns is $2T = C + x + y$. Hence $x + y = 2T - C = 2 \times 17 - 13 = 21$, which is impossible since $x + y$ is at most $9 + 10$.
However, when $T = 18$ we have $C = 17$ and $x + y = 19$ so that x and y are 9 and 10. One possible solution in this case is

2	8	3	5
10			9
6	1	7	4

Hence the minimum possible value of T is 18.