Solutions to the Olympiad Hamilton Paper

1. The required number is divisible by 12, so it is also divisible by 3 and by 4.

   Since the number is divisible by 3 the sum of its digits is a multiple of 3. But the digits are all 0 or 1, hence at least three of the digits are 1.

   Since the number is divisible by 4 the only possible final two digits are 00.

   Hence the smallest number of the required form is 11100.

2. We know that $\angle ABC = 90°$ and the sum of the angles in triangle ABC is 180°. Therefore

$$\angle QBC = 90° - \angle ABQ. \qquad (1)$$

   Similarly, in triangle BPQ, $\angle BPQ = 90°$ so that

$$\angle BQC = 90° - \angle QBP. \qquad (2)$$

   But $\angle ABQ = \angle QBP$ and hence from (1) and (2) $\angle QBC = \angle BQC$.

   Thus the triangle QBC is isosceles, with $CB = CQ$.

3. Let the cost of a Fudge Bar be F pence, the cost of a Sparkle be S pence and the cost of a Chomper be C pence. Then,

$$4F + S + C = 100 \qquad (1)$$

$$2F + S + 3C = 70 \qquad (2)$$

$$F + 2S = 50. \qquad (3)$$

   Subtracting equation (2) from three times equation (1) gives

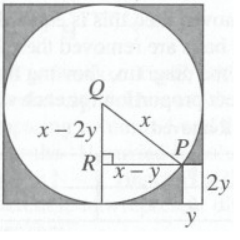
$$10F + 2S = 230. \qquad (4)$$

   Now subtracting equation (4) from ten times equation (3) gives $18S = 270$, and hence $S = 15$.

   Thus a Sparkle costs 15p. (A Fudge Bar costs 20p and a Chomper costs 5p.)

4. Let x be the radius of the circle so that the square has side $2x$.

   Let y be the width of the rectangle so that the height of the rectangle is $2y$.



   Applying Pythagoras' theorem to the triangle PQR, we have $x^2 = (x - 2y)^2 + (x - y)^2$. Rearranging gives $x^2 - 6xy + 5y^2 = 0$, which factorises to give $(x - y)(x - 5y) = 0$. Hence $x = y$ or $x = 5y$.

   However, $x = y$ is impossible since there would then be no circle.

   When $x = 5y$ the ratio of the area of the square to the area of the rectangle is $(10y)^2 : 2y^2$ which simplifies to $50 : 1$.

5. Let p be the number of pupils initially in the Academy.
Let t be the number of teachers initially in the Academy.
Therefore initially the number of pupils per teacher is p/t.
If 10 new teachers are employed, the number of pupils per teacher is $p/(t + 10)$. Hence

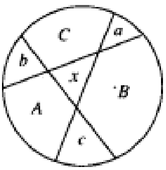
$$\frac{p}{t + 10} = \frac{p}{t} - 5. \qquad (1)$$

If 20 new teachers are employed the number of pupils per teacher is $p/(t + 20)$. Hence

$$\frac{p}{t + 20} = \frac{p}{t} - 8. \qquad (2)$$

From (1), $p = \frac{1}{2}t(t + 10)$. (3)
From (2), $p = \frac{2}{5}t(t + 20)$. (4)
Equating (3) and (4) gives $5t(t + 10) = 4t(t + 20)$, hence $t(t - 30) = 0$.
But $t > 0$ so that $t = 30$, and substituting $t = 30$ into (3) or (4) gives $p = 600$.
Thus there are 600 pupils at Abertawe Academy.

6. Label the numbers in each region a, b, c, A, B, C, as shown.



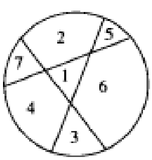
Since $1 + 2 + \ldots + 6 + 7 = 28$, the total of the numbers in the circle on each side of one of the chords is equal to 14. Hence

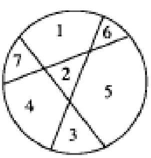
$$a + B + c = 14$$

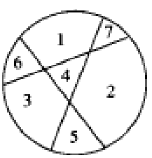
and $$A + x + B + c = 14$$

from which we deduce that $a = A + x$. Similarly, $b = B + x$ and $c = C + x$. Therefore three of the numbers in the circle, a, b and c, are greater than x. It follows that x is at most 4.

The following diagrams show that solutions are possible with $x = 1, 2$ or 4.







When $x = 3$, we have $a = A + 3$, $b = B + 3$ and $c = C + 3$. The possible pairs of values for (a, A), (b, B) and (c, C) are (4, 1), (5, 2), (6, 3) and (7, 4). The pair (6, 3) is not allowed, since $x = 3$. Finally, since the three remaining pairs, (4, 1), (5, 2) and (7, 4), do not include 6, there is no solution when $x = 3$.
Thus there are three possible values for x.