# Solutions to the Olympiad Hamilton Paper

1. Let the number of consecutive integers which sum to 75 be $n$. We consider separately the cases when $n$ is odd and when $n$ is even.

   If $n$ is odd, let the middle number be $k$. So the numbers are $k - \frac{1}{2}(n - 1), \ldots, k - 2, k - 1, k, k + 1, k + 2, \ldots, k + \frac{1}{2}(n - 1)$. The sum of these numbers is $nk$, so $nk = 75$. [*Alternatively*, this equation may be derived by observing that the middle number, $k$, is the average of all the numbers, $\frac{75}{n}$.] As $n$ is greater than 1 and odd, the possible values of $(n, k)$ are (3,25), (5,15), (15,5), (25,3) and (75,1).

   If $n = 3$, we have $24 + 25 + 26 = 75$. If $n = 5$, we have $13 + 14 + 15 + 16 + 17 = 75$. If $n = 15$, we have $(-2) + (-1) + 0 + 1 + 2 + 3 + \ldots + 12 = 75$, but not all of these are positive integers. Similarly, when $n = 25$ and when $n = 75$, the sequence contains negative integers. So there are two different ways for which the number of consecutive integers is odd.

   If $n$ is even, let the middle numbers be $k - 1$ and $k$. So the numbers are $k - \frac{1}{2}n, \ldots, k - 2, k - 1, k, k + 1, k + 2, \ldots, k + (\frac{1}{2}n - 1)$. The sum of these numbers is $nk - \frac{1}{2}n$, so $n(k - \frac{1}{2}) = 75$, that is $n(2k - 1) = 150$. [*Alternatively*, this equation may be derived by observing that the average of the middle two numbers, $\frac{1}{2}\{k + (k - 1)\}$, is the average of all the numbers, $\frac{75}{n}$.] As $n$ is greater than 1 and even, the possible values of $(n, k)$ are (2, 38), (6, 13), (10, 8), (30, 3), (50, 2) and (150,1).

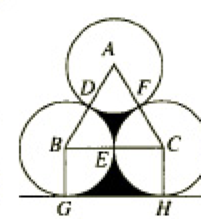
   If $n = 2$, we have $37 + 38 = 75$. If $n = 6$, we have $10 + 11 + 12 + 13 + 14 + 15 = 75$. If $n = 10$, we have $3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 = 75$. When $n = 30$, when $n = 50$ and when $n = 150$, the sequence contains negative integers. So there are three different ways for which the number of consecutive integers is even, giving five different ways overall.

   *Alternatively*, let the first integer be $f$ and let the number of integers be $n$. Then the sum of the integers is $f + (f + 1) + (f + 2) + \ldots + (f + n)$, which is $nf + (1 + 2 + 3 + \ldots + n)$, or $nf + T_{n-1}$, where $T_{n-1}$ is the $(n - 1)$th triangular number. The values of $n$ and $T_{n-1}$ are given in the table below, along with the solution, $f$, of the corresponding equation $nf + T_{n-1} = 75$, in the cases when this solution is an integer. For example, when $n = 5$, solving $6f + 15 = 75$ gives $f = 10$, whereas when $n = 7$, solving $7f + 21 = 75$ gives $f = \frac{54}{7}$.

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{n-1}$ | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 45 | 55 | 66 |
| $f$ | 37 | 24 | – | 13 | 10 | – | – | – | 3 | – | – |

   Note that $T_{n-1} > 75$ for $n > 12$, so no other solutions are possible, since we require $f > 0$. Hence there are five different solutions for $f$ and therefore five different ways of expressing 75 as the sum of two or more consecutive positive integers.

2. Let $A$, $B$, $C$ be the centres of the circles and let $D$, $E$, $F$ be the points where the circles touch each other. Let $G$, $H$ be the points where the tangent touches two of the circles.

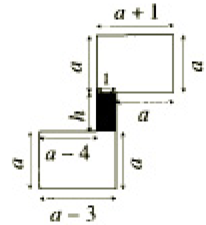



   As each circle has radius 1, triangle $ABC$ is an equilateral triangle of side 2. So each arc $DE$, $EF$, $FD$ has length $\frac{1}{6} \times 2\pi \times 1 = \frac{\pi}{3}$.

   Consider quadrilateral $BCHG$. Both $BG$ and $CH$ are radii, so they have length 1 and are perpendicular to tangent $GH$. Hence $BCHG$ is a rectangle. So each arc $GE$, $HE$ has length $\frac{1}{4} \times 2\pi \times 1 = \frac{\pi}{2}$.

   Now the perimeter of the shaded region consists of arcs $DE$, $EF$, $FD$, $GE$, $HE$ and line segment $GH$, which has length 2. The required perimeter is $3 \times \frac{\pi}{3} + 2 \times \frac{\pi}{2} + 2 = 2\pi + 2$.

3. Let $h$ cm be the distance shown in the diagram, which omits the units for clarity.




Then from the perimeter of the figure, we have $8a + 2h - 6 = 72$. Hence $h = 39 - 4a$.

From the area of the figure, we have $(a + 1) \times a + 1 \times h + (a - 3) \times a = 147$. Substituting for $h$ gives

$$a^2 + a + 39 - 4a + a^2 - 3a = 147$$

i.e. $$2a^2 - 6a - 108 = 0$$

i.e. $$a^2 - 3a - 54 = 0$$

i.e. $$(a - 9)(a + 6) = 0$$

Hence $a = 9$ or $a = -6$, but the latter is not possible as $a$ is positive. So $a = 9$.

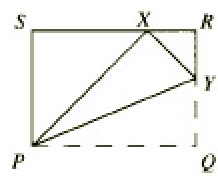
4. Clearly there are no single-digit 'unfortunate' numbers.

Let '$ab$' be a two-digit unfortunate number. Then $10a + b = 13(a + b)$, that is $3a + 12b = 0$. This is impossible as $a > 0$ and $b \geqslant 0$. So there are no two-digit unfortunate numbers.

Let '$abc$' be a three-digit unfortunate number. Then $100a + 10b + c = 13(a + b + c)$, that is $87a = 3b + 12c$, which simplifies to $29a = b + 4c$. We deduce that the only possible value of $a$ is 1, since otherwise $29a \geqslant 58$, whereas the maximum possible value of $b + 4c$ is 45. The equation $b + 4c = 29$ has exactly three (non-negative integer) solutions: $(b, c) = (1, 7), (5, 6)$ or $(9, 5)$. So 117, 156 and 195 are the only three-digit unfortunate numbers.

Let '$abcd$' be a four-digit unfortunate number. Then $1000a + 100b + c + d = 13(a + b + c + d)$. Now the minimum value of the left-hand side of this equation is 1000, but the maximum value of the right-hand side is $13 \times 36$, which is 468. So there are no four-digit unfortunate numbers. Adding more digits makes this imbalance worse, since each new digit contributes more than an extra 1000 to the left-hand side but at most an extra $13 \times 9$, that is 117, to the right-hand side. Hence there are no unfortunate numbers which have more than four digits. So the complete list of all unfortunate numbers is 117, 156, 195.

5. As $PY$ is the fold line, triangle $PXY$ is the reflection of triangle $PQY$ in line $PY$. Hence $PX = PQ = \sqrt{2}$. Applying Pythagoras' theorem to triangle $PSX$ gives




$$SX = \sqrt{PX^2 - PS^2} = \sqrt{2 - 1} = 1.$$

So $RX = \sqrt{2} - 1$. Also, since $SX = SP = 1$, triangle $PSX$ is an isosceles right-angled triangle and $\angle SXP = 45°$.

Now angle $PXY$ is the image under reflection of angle $PQY$ and is therefore a right angle. So $\angle RXY = 180° - (90° + 45°) = 45°$ and we deduce that triangle $RXY$ is also an isosceles right-angled triangle. So $RY = RX = \sqrt{2} - 1$.

It remains to find the length of $XY$. As $QY$ folds onto $XY$, we deduce that

$$XY = QY = QR - RY = 1 - \left(\sqrt{2} - 1\right) = 2 - \sqrt{2}.$$

6. Let the heights in cm of Anna, Bob, Claire and Duncan be $a$, $b$, $c$, $d$ respectively. Then $a = c + 8$ and $b = d - 4$. We know that neither Anna nor Duncan is the shortest of the four and also that neither Claire nor Bob is the tallest. So the possible pairings of the tallest and shortest of the friends are Anna and Bob, Anna and Claire, Duncan and Bob, Duncan and Claire.

   Case (i): Anna is the tallest and Bob is the shortest. This gives the equation $a + b = c + d + 2$. Substituting for $a$ and $b$ gives $c + 8 + d - 4 = c + d + 2$, which is not possible.

   Case (ii): Anna is the tallest and Claire is the shortest. This gives the equation $a + c = b + d + 2$. Also, $a + b + c + d = 672$, so we may deduce that $b + d = 335$ and that $a + c = 337$. Hence $a = 172\frac{1}{2}$, $b = 165\frac{1}{2}$, $c = 164\frac{1}{2}$, $d = 169\frac{1}{2}$. These values are consistent with all the information given.

   Case (iii): Duncan is the tallest and Bob is the shortest. This gives the equation $d + b = a + c + 2$. Also, $a + b + c + d = 672$, so we may deduce that $b + d = 337$ and that $a + c = 335$. Hence $a = 171\frac{1}{2}$, $b = 166\frac{1}{2}$, $c = 163\frac{1}{2}$, $d = 170\frac{1}{2}$. However, these values are not consistent with Duncan being the tallest and Bob the shortest, so a contradiction exists and this case is not possible.

   Case (iv): Duncan is the tallest and Claire is the shortest. This gives the equation $c + d = a + b + 2$. Substituting for $a$ and $b$ gives $c + d = c + 8 + d - 4 + 2$, which is not possible.

   So only one of the four cases leads to values of $a$, $b$, $c$, $d$ which are consistent with the information given and we conclude that the heights of Anna, Bob, Claire and Duncan are 1 m $72\frac{1}{2}$ cm, 1 m $65\frac{1}{2}$ cm, 1 m $64\frac{1}{2}$ cm, 1 m $69\frac{1}{2}$ cm, respectively.

   *Alternatively*, let the heights in cm of Anna, Bob, Claire and Duncan be $a$, $b$, $c$, $d$ respectively. From the information in the question,

$$a + b + c + d = 672$$

$$a - c = 8$$

$$d - b = 4.$$

   Adding these equations gives

$$2(a + d) = 684.$$

   Hence $a + d = 342$ and $b + c = 330$. Now let the heights of the tallest and shortest be $t$ cm and $s$ cm, respectively. Then, from the given information, $(t + s) + (t + s - 2) = 672$, so that $t + s = 337$. One of Anna and Duncan is the tallest, so that $a$, $d$ are $t$, $342 - t$ in some order. Similarly, $b$, $c$ are $s$, $330 - s$ in some order, that is, $337 - t$ and $t - 7$.

   We therefore now know that subtracting one of $337 - t$, $t - 7$ from $t$ gives either $a - c$ or $b - d$, that is, 8 or 4. Noting that $t - (t - 7) = 7$, we deduce that $t - (337 - t)$ is 8 or 4. Hence $2t - 337 = 8$ or $2t - 337 = 4$ and so $t = \frac{341}{2}$ or $\frac{345}{2}$. But in the first case, $t$ is less than $342 - t$, which is not possible. In the second case, the heights of Anna, Bob, Claire and Duncan are 1 m $72\frac{1}{2}$ cm, 1 m $65\frac{1}{2}$ cm, 1 m $64\frac{1}{2}$ cm, 1 m $69\frac{1}{2}$ cm, respectively.