2014

Solutions to the Olympiad Cayley Paper

C1. The two-digit integer ‘19’ is equal to the product of its digits (1×9) plus the sum of its digits ($1 + 9$). Find all two-digit integers with this property.

Solution

If such a two-digit number has first digit a and second digit b, then its value is $10a + b$. The given condition then says that the product ab of the digits, plus the sum $a + b$ of the digits, is equal to $10a + b$, in other words,

$$ab + a + b = 10a + b.$$

Subtracting b from both sides, we obtain

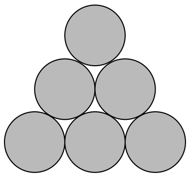
$$ab + a = 10a.$$

Since a two-digit number cannot have first digit zero, we have $a \neq 0$ and we can divide both sides by a to get $b + 1 = 10$, that is, $b = 9$. This shows that a number with this property has second digit 9.

We therefore check the numbers 19, 29, 39, …, 89 and 99, and find that all of them have the required property.

C2. Six pool balls numbered 1−6 are to be arranged in a triangle, as shown. After three balls are placed in the bottom row, each of the remaining balls is placed so that its number is the difference of the two below it.

Which balls can land up at the top of the triangle?



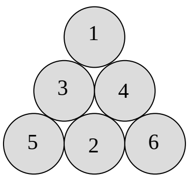
Solution

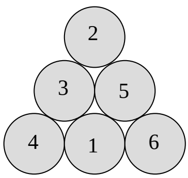
First we observe that the ball numbered 6 is on the bottom row of the triangle, since there are no permitted numbers which differ by 6 (because the furthest apart are 1 and 6 itself, and they differ by only 5).

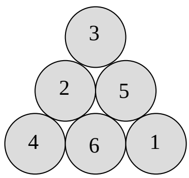
This tells us not only that 6 cannot appear at the top, but also that 5 cannot. Indeed, if 5 is at the top, then the middle row is 1 and 6 in some order, and that means 6 is not on the bottom row.

If 4 is at the top, then the numbers below are either 2 and 6 (which, as above, is not permitted), or 1 and 5. In the latter case, the numbers below the 5 have to be 1 and 6, but we have already used the 1, so this cannot happen.

That just leaves three possibilities for the top number: 1, 2 and 3. The following examples show that they can all be achieved.







C3. Rachel gave half of her money to Howard. Then Howard gave a third of all his money to Rachel. They each ended up with the same amount of money.

Find the ratio

amount that Rachel started with : amount that Howard started with.

Solution

Suppose Howard starts with h pence and Rachel with r pence. Then Rachel gives $\frac{r}{2}$ to Howard; so after this Howard has $h + \frac{r}{2}$.

Next, Howard gives one third of his money to Rachel, so he has two thirds left. Thus he now has

$$\frac{2}{3}\left(h + \frac{r}{2}\right), \text{ that is, } \frac{2h}{3} + \frac{r}{3}.$$

We are told that they then have the same amount of money, and so each of them has half the total amount. Therefore

$$\frac{2h}{3} + \frac{r}{3} = \frac{h}{2} + \frac{r}{2}.$$

Multiplying both sides by 6, we get

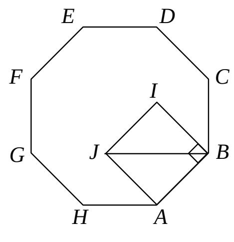
$$4h + 2r = 3h + 3r,$$

and subtracting $3h + 2r$ from both sides we get $h = r$ or, in words, Howard and Rachel started with the same amount of money. So the ratio we were asked to find is 1 : 1.

C4. The square *ABIJ* lies *inside* the regular octagon *ABCDEFGH*. The sides of the octagon have length 1.

Prove that $CJ = \sqrt{3}$.

Solution



The exterior angles of any polygon add up to 360°, so for a regular octagon they are 45° each. That means the interior angles are 180° – 45° = 135° each.

Since the angle *ABI* is a right angle, the angle *IBC* is $135° - 90° = 45°$.

Consider now the line *JB*.

This is the diagonal of the unit square *ABIJ*, and so

$$JB^2 = 1^2 + 1^2 = 2,$$

by Pythagoras' theorem for the triangle *ABJ*.

Also, $\angle JBI = 45°$, so $\angle JBC = \angle JBI + \angle IBC = 45° + 45° = 90°$.

This means that triangle *JBC* is right-angled at *B*. We have computed JB^2 and we know $BC = 1$, so, applying Pythagoras' theorem to triangle *JBC*, we now get

$$CJ^2 = JB^2 + 1^2$$
$$= 2 + 1,$$

that is, $CJ = \sqrt{3}$, as required.

C5. Four types of rectangular tile have sizes 300 mm × 300 mm, 300 mm × 600 mm, 600 mm × 600 mm and 600 mm × 900 mm. Equal numbers of each type of tile are used, without overlaps, to make a square.

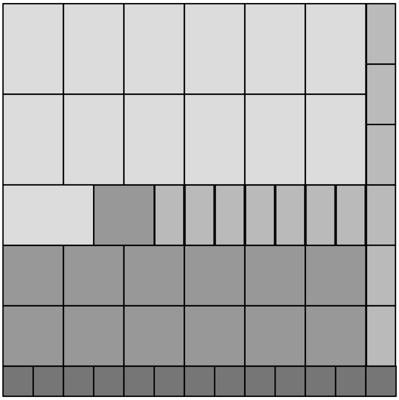
What is the smallest square that can be made?

Solution

Let us say that one unit is 300 mm, so that the permitted tiles are 1 × 1, 1 × 2, 2 × 2 and 2 × 3.

Since the sides of all the tiles have lengths that are a whole number of units, any square made out of them will have sides of length N that is a whole number of units. This square has area N^2.

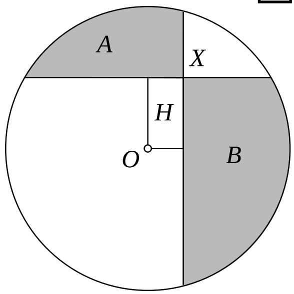
Also, the total area of one tile of each type is $1 + 2 + 4 + 6 = 13$, so N^2 is a multiple of 13. The smallest such N is 13 itself, so we ask ourselves if such a tiling is possible with 13 copies of each tile. It can indeed be done, as the example in the figure shows.



This square, which we have proved to be the smallest possible, measures 13 units on each side, or 3.9 m × 3.9 m.

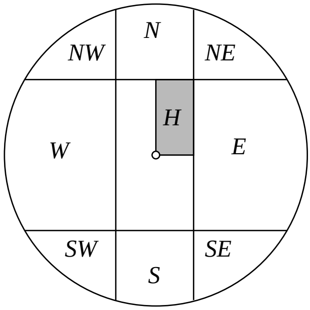
C6. A couple own a circular piece of land that has area 2500 m^2. The land is divided into four plots by two perpendicular chords that intersect at *X*. Their rectangular house *H* has diagonally opposite corners at *X* and at the centre of the circle *O*, as shown. The two plots *A* and *B* have a combined area of 1000 m^2.

What is the area occupied by the house?



Solution

Suppose the couple construct two new fences, by reflecting the two given perpendicular chords in the lines through the walls of their house.



This creates a symmetric configuration, as shown, so that their piece of land now consists of:

(i) one central plot, containing the couple's house;

(ii) two plots *W* and *E*, of the same size and shape, at the west and east;

(iii) two plots *N* and *S*, of the same size and shape, at the north and south;

(iv) four plots *NW*, *NE*, *SW* and *SE*, all of the same size and shape, in each corner, at the northwest, northeast, southwest and southeast.

The original two plots *A* and *B*, with the combined area 1000 m^2, now consist of *N* and *NW*, and *E* and *SE*.

But those areas are equal, respectively, to *S* and *SW*, and *W* and *NE*, so those have a combined area of 1000 m^2. That means that all the regions, apart from the central one, have a combined area of 2000 m^2, leaving 500 m^2 inside the central region.

Of this, one quarter is the couple's house, since the couple's house consists of everything in the central region northeast of the centre of the circle. So their house occupies 125 m^2.