Solutions to the Olympiad Cayley Paper

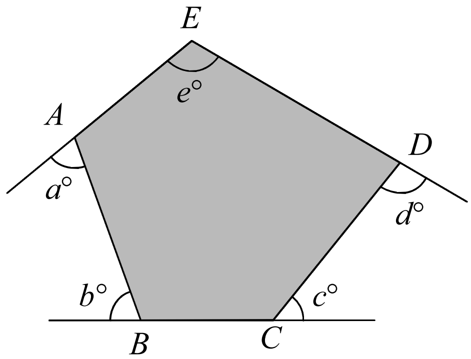
1. What is the smallest non-zero multiple of 2, 4, 7 and 8 which is a square?

Solution

The conditions state that the required number is a multiple of 2^1, of 2^2, of 7^1 and of 2^3, respectively. This means that it is a multiple of $2^3 \times 7^1$. Therefore we need to find the smallest non-zero multiple of $2^3 \times 7^1$ which is a square.

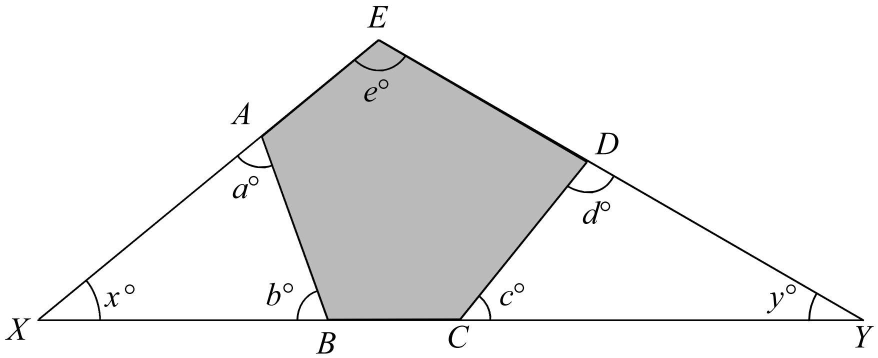
In the prime factorisation of a square number, each prime can only occur an even number of times. Thus the smallest non-zero multiple of $2^3 \times 7^1$ that is also a square is $2^4 \times 7^2$, which is 784.

2. The diagram shows a pentagon *ABCDE*.
Prove that $a + b + c + d = 180 + e$.



Solution

Many different solutions are possible; we give just one.



We extend sides, as shown in the figure: let lines *EA*, *CB* meet at *X*, lines *ED*, *BC* meet at *Y* and let $x°$, $y°$ be the angles of triangle *EXY* at *X* and *Y* respectively.

Applying 'angles in a triangle add to 180°' to the triangles *AXB*, *CYD* and *XYE*, we get the equations

$$x + a + b = 180,$$

$$y + c + d = 180,$$

and $$x + y + e = 180.$$

Adding the first two equations and subtracting the third, we obtain

$$a + b + c + d - e = 180.$$

Therefore

$$a + b + c + d = 180 + e$$

as required.

3. Consider sequences of positive integers for which both the following conditions are true:

(a) each term after the second term is the sum of the two preceding terms;
(b) the eighth term is 260.

How many such sequences are there?

Solution

Let a and b be the first and second terms respectively. Then the first eight terms are

$$a,\ b,\ a+b,\ a+2b,\ 2a+3b,\ 3a+5b,\ 5a+8b,\ 8a+13b.$$

Hence we seek solutions to the equation

$$8a + 13b = 260, \qquad (1)$$

where a and b are positive integers, since any such solution will generate a sequence of positive integers of the required sort.

Now $8a = 260 - 13b = 13(20 - b)$, which is a multiple of 13. Therefore $8a$ is a multiple of 13, so that a is a multiple of 13.

Let $a = 13k$, where k is a positive integer. Then equation (1) becomes

$$8 \times 13k + 13b = 260,$$

so that

$$8k + b = 20.$$

Since b and k are positive integers there are therefore only two possible values for k, namely 1 and 2. When $k = 1$, we have $a = 13$ and $b = 12$. When $k = 2$, we have $a = 26$ and $b = 4$. Hence there are two possible sequences.

4. The positive integer m has leading digit 1. When this digit is moved to the other end, the result is $3m$. What is the smallest such m?

Solution

Method 1

It is useful to find what happens when the digits from 0 to 9 are multiplied by 3. The tables show the resulting last digits.

Digit d	Last digit of $3 \times d$
0	0
1	3
2	6
3	9
4	2

Digit d	Last digit of $3 \times d$
5	5
6	8
7	1
8	4
9	7

We know that $3m$ ends in a 1. From the table, the only digit that can be multiplied by 3 to give a units digit of 1 is 7. Thus m ends in 7 and so $3m$ ends in 71.

Hence we now wish to find a two-digit number 'a7' which can be multiplied by 3 to give something ending in 71. Since $3 \times 7 = 21$ and $71 - 21 = 50$, we require $3 \times a$ to end in 5, and from the table we deduce that $a = 5$. Thus m ends in 57, and $3 \times 57 = 171$.

Continuing, we wish to find a three-digit number ending in 'b57' which can be multiplied by 3 to give something ending in 571. In a similar way, we deduce that the only answer is 857, and $3 \times 857 = 2571$.

Similarly, when we look for a four-digit number ending in 857 which can be multiplied by 3 to give something ending in 8571, we find only 2857, and $3 \times 2857 = 8571$.

When we look for a five-digit number ending in 2857 which can be multiplied by 3 to give something ending in 28571, we find $3 \times 42\,857 = 428\,571$.

So m is 142 857, and indeed when we move the 1 to the end we get $428\,571 = 3 \times 142\,857$.

Method 2

The integer m clearly has more than one digit.

Suppose that m has two digits, so that $m =$ '1a' and $3m =$ 'a1'. But '1a' $= 10 + a$ and 'a1' $= 10a + 1$. Therefore we have

$$3(10 + a) = 10a + 1,$$

that is

$$29 = 7a$$

which has no solutions since a is an integer.

Suppose that m has three digits, so that $m =$ '1ab'. In a similar way, we obtain

$$3\left(100 + \text{`}ab\text{'}\right) = 10 \times \text{`}ab\text{'} + 1,$$

that is

$$299 = 7 \times \text{`}ab\text{'},$$

which also has no solutions since 'ab' is an integer.

Similarly, we find that m cannot have four digits, because $2999 = 7 \times$ 'abc' has no solutions. Five digits also fails, because 29 999 is not a multiple of 7. But 299 999 is $7 \times 42\,857$. Thus the smallest such m has six digits and is 142 857.

5. Pablo plans to take several unit cubes and arrange them to form a larger cube. He will then paint some of the faces of the larger cube. When the paint has dried, he will split the larger cube into unit cubes again.

Suppose that Pablo wants exactly 150 of the unit cubes to have no paint on them at all. How many faces of the larger cube should he paint?

Solution

Suppose that the larger cube is made up of $n \times n \times n$ smaller cubes.

Now the unpainted smaller cubes form a cuboid, and we can work out the number of unpainted smaller cubes in the form abc. Here a is n (if neither the left nor the right face is painted), or $n - 1$ (if exactly one of the two faces is painted), or $n - 2$ (if both are). Similarly, b is n, $n - 1$ or $n - 2$ depending on which of the front and back faces is painted, and c is n, $n - 1$ or $n - 2$ depending on which of the top and bottom faces are painted.

So we are trying to express 150 as a product of three numbers that differ by at most 2.

Now the prime factorisation of 150 is $2 \times 3 \times 5 \times 5$. Combining the 2 and 3 gives $150 = 5 \times 5 \times 6$, which is of the desired form. Any other option has the form $150 = 1 \times r \times s$, or $150 = 2 \times r \times s$ or $150 = 3 \times r \times s$, and none of these is possible since one of r and s will differ from the first factor by more than 2 (indeed, one of r and s will be greater than 7, because $3 \times 7 \times 7 < 150$). Thus $5 \times 5 \times 6$ is the only way of writing 150 in the required form.

Finally, $abc = 5 \times 5 \times 6$ corresponds to either

$$abc = (n - 1) \times (n - 1) \times n \text{ with } n = 6,$$

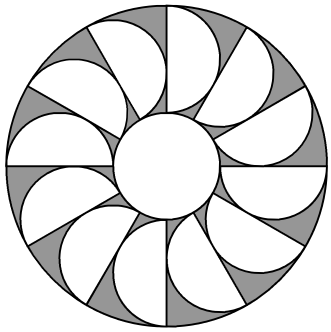
or

$$abc = (n - 2) \times (n - 2) \times (n - 1) \text{ with } n = 7.$$

These in turn correspond to either two or five faces being painted: if he has made a $6 \times 6 \times 6$ cube he should paint two (adjacent) faces; if he has made a $7 \times 7 \times 7$ cube he should paint five of the six faces.

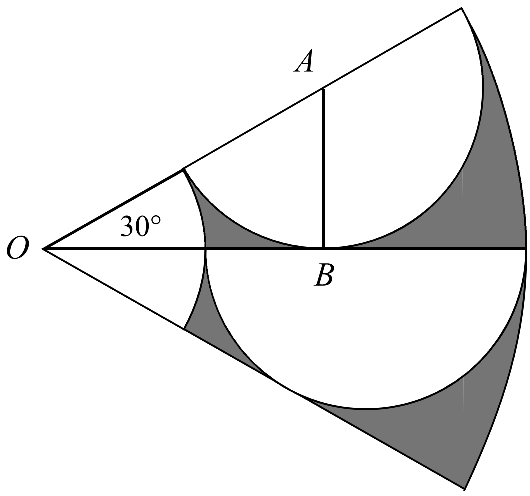
6. The diagram shows an *annulus*, which is the region between two circles with the same centre. Twelve equal touching semicircles are placed inside the annulus. The diameters of the semicircles lie along diameters of the outer circle.

   What fraction of the annulus is shaded?



Solution

The twelve diameters of the semicircles are regularly spaced: each is 30° from the next. Consider two adjacent semicircles (see the figure). Let *A* be the centre of the first, and let *B* be the point of tangency of the first semicircle with the diameter of the second. Finally, let *O* be the centre of the annulus.



Angle *ABO* is a right angle, since *OB* is a tangent and *AB* is a radius. Thus angle *OAB* is 60°, because the angles in triangle *OAB* sum to 180°.

If we let $AB = 1$, then $OA = 2$ (this is a famous property of a 30°, 60°, 90° triangle, coming from considering it as half of an equilateral triangle).

This means that the outer radius of the annulus is 3 and the inner radius is 1, so the annulus has area $\pi \times 3^2 - \pi \times 1^2 = 8\pi$. The semicircles have radius 1, so twelve of them have total area $12 \times \frac{1}{2} \times \pi \times 1^2 = 6\pi$. Therefore the shaded area in the given diagram is $8\pi - 6\pi = 2\pi$.

Hence $\frac{1}{4}$ of the annulus is shaded.