Solutions to the Olympiad Cayley Paper

1. The digits *p*, *q*, *r*, *s* and *t* are all different.

   What is the smallest five-digit integer '*pqrst*' that is divisible by 1, 2, 3, 4 and 5?

   Solution

   Note that all five-digit integers are divisible by 1.

   Next, notice that if a number is divisible by 2 and 5, then it is divisible by 10 and hence has last digit *t* equal to 0. Since all the digits are different, that means all the other digits are non-zero.

   Among five-digit numbers, those that begin with 1 are smaller than all those which do not. So if we find a number of the required form with first digit 1, then it will be smaller than numbers with larger first digits.

   Similarly, those with first two digits 12 are smaller than all other numbers with distinct non-zero digits. And, in fact, those with first three digits 123 are smaller than all others. Hence if we find such a number with the required properties, it will be smaller than all others.

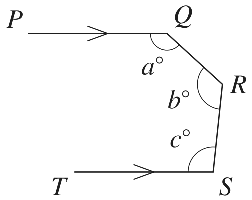
   So let us try to find a number of the form '123*s*0'.

   A number is a multiple of four only if its last two digits form a multiple of four. So we need consider only the case where *s* is even.

   Similarly, a number is a multiple of three if and only if the sum of its digits is a multiple of three. Since $1 + 2 + 3 + s + 0 = 6 + s$, we only need consider the case where *s* is a multiple of three.

   Thus 12360 is the only number of the form '123*s*0' which is divisible by 1, 2, 3, 4 and 5, and as we have explained along the way, it is the smallest number with the required divisibility properties.

2. In the diagram, PQ and TS are parallel.
Prove that $a + b + c = 360$.

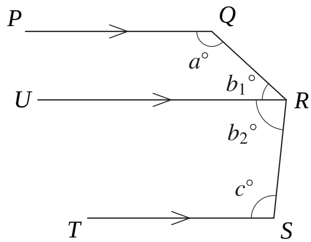


Solution

We describe just two of the many different methods that are possible.

Method 1

Draw a line UR parallel to PQ and TS passing through R. This line divides the angle $b°$ into two parts; suppose these parts have sizes $b_1°$ and $b_2°$, as shown.



Then angles $a°$ and $b_1°$ are supplementary (because PQ and UR are parallel), as are angles $b_2°$ and $c°$. Therefore

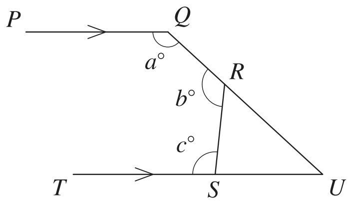
$$a + b_1 = 180$$

and $$b_2 + c = 180.$$

By adding these two equations, we obtain the desired result since $b_1 + b_2 = b$.

Method 2

Extend QR and TS until they meet at a new point U.



Then angles $\angle PQR$ and $\angle SUR$ are supplementary (because PQ and TU are parallel), that is, $a° + \angle SUR = 180°$.

However, we have

$$\angle SUR + \angle URS + \angle RSU = 180°$$

by properties of angles in a triangle. This means that $a° = \angle URS + \angle RSU$.

We also have $b° + \angle URS = 180° = c° + \angle RSU$ since they are angles on a straight line.

Putting all this together we get

$$a° + b° + c° = \angle URS + \angle RSU + (180° - \angle URS) + (180° - \angle RSU) = 360°,$$

as needed.

3. Three loaves of bread, five cartons of milk and four jars of jam cost £10.10.

Five loaves of bread, nine cartons of milk and seven jars of jam cost £18.20.

How much does it cost to buy one loaf of bread, one carton of milk and one jar of jam?

Solution

We write B for the price of a loaf of bread in pence, M for the price of a carton of milk in pence, and J for the price of a jar of jam in pence. Then we can interpret the given information as saying

$$3B + 5M + 4J = 1010$$

and $$5B + 9M + 7J = 1820.$$

If we double the first equation we get

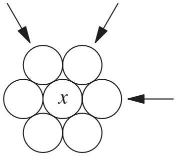
$$6B + 10M + 8J = 2020,$$

and if we then subtract the second equation from this we get

$$B + M + J = 200,$$

or, in words, the total cost of a loaf of bread, a carton of milk and a jar of jam is £2.

4. The diagram shows seven circles. Each of the three arrows indicates a ‘line of three circles’.

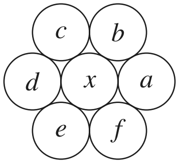


The digits from 1 to 7 inclusive are to be placed in the circles, one per circle, so that the sum of the digits in each of the three indicated ‘lines of three circles’ is the same.

Find all possible values of x.

Solution

Let us write a, b, c, d, e and f for the six values in the outer circles, as shown, and write s for the common sum of the three lines.



The given conditions then say that

$$a + x + d = s,$$

$$b + x + e = s,$$

and $$c + x + f = s.$$

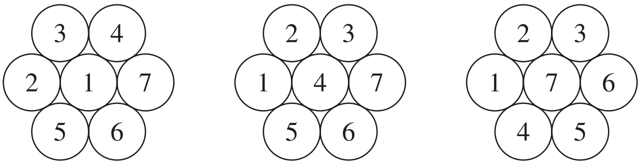
If we add these equations together, we get

$$a + b + c + d + e + f + 3x = 3s.$$

However, the sum $a + b + c + d + e + f + x$ is equal to 28, since the numbers are 1, 2, 3, 4, 5, 6 and 7 in some order. So, rewriting, we get

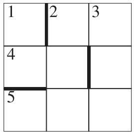
$$2x + 28 = 3s,$$

which says that $2x + 28$ is a multiple of 3. We can quickly check that this happens only when $x = 1,\ 4$ or 7. Hence all other values of x are impossible. Lastly, we demonstrate with three examples that these three values of x are all possible:



5. Every cell of the following crossnumber is to contain a single digit. No clue has an answer starting with zero.

Prove that there is exactly one solution to the crossnumber.



Across

2 Sum of the digits of 2 Down.

4 Prime.

5 1 Down + 2 Across + 3 Down.

Down

1 Product of two primes.

2 Multiple of 99.

3 Square of 4 Across.

Solution

Firstly, the answer to 2 DOWN has to be one of 198, 297, 396, 495, 594, 693, 792, 891 or 990. In all of these cases, the sum of the digits is the same: the answer to 2 ACROSS is thus 18.

Knowing the first digit is 1 immediately allows us to narrow down 2 DOWN to 198. Also, 3 DOWN begins with an 8 and is the square of 4 ACROSS, an integer which ends in 9. The number $19^2 = 361$ is too small, while 39^2 and subsequent squares have four or more digits. Hence 4 ACROSS is 29 and 3 DOWN is 841.

We now use the clue for 5 ACROSS: we get

$$\text{`}x2\text{'} + 18 + 841 = \text{`}y81\text{'},$$

where x is the missing top left-hand digit and y is the missing bottom left-hand digit. Clearly y is 8, since even if x were 9 it would not be big enough to give a total of 981.

We can then subtract to find the answer to 1 DOWN: it is given by

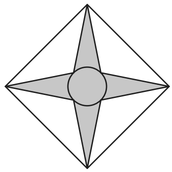
$$881 - 841 - 18 = 22.$$

We have uniquely identified values for all nine digits, using the clues.

2	1	8
2	9	4
8	8	1

In checking, we observe that we have not used the clue for 1 DOWN; however $22 = 2 \times 11$, so it does indeed work.

6. The diagram shows a symmetrical four-pointed star. Four vertices of the star form a square and the other four vertices lie on a circle. The square has sides of length $2a$ cm. The shaded area is one third of the area of the square.



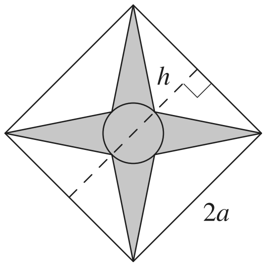
What is the radius of the circle?

Solution

Method 1

Since the shaded area is one third of the area of the square, the area not shaded is two thirds. So each white triangle has an area of

$$\tfrac{1}{4} \times \tfrac{2}{3} \times 4a^2 = \tfrac{2}{3}a^2.$$

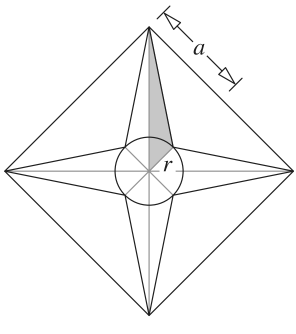


However, the base of a white triangle is $2a$, so the height h cm is given by $\tfrac{1}{2} \times 2a \times h = \tfrac{2}{3}a^2$. Hence $h = \tfrac{2}{3}a$.

Now the dashed line in the figure comprises two of these heights and a diameter of the circle. But the dashed line has a length of $2a$, the length of a side of the square. Thus the radius of the circle is $\tfrac{1}{3}a$.

Method 2

We first cut the star into eight congruent triangles, as shown.



Each triangle may be considered to have base r, the radius of the circle, and height a, as indicated for the shaded triangle.

Using the formula for the area of a triangle, and the fact that these eight triangles together have area one third of the area of the whole square, we get

$$8 \times \tfrac{1}{2} \times r \times a = \tfrac{1}{3} \times 4a^2,$$

which we solve to obtain $r = \tfrac{1}{3}a$.