Solutions to the Olympiad Cayley Paper

1 A palindromic number is one which reads the same when its digits are reversed, for example 23832.

What is the largest six-digit palindromic number which is exactly divisible by 15?

Solution

We note that being divisible by 15 is the same as being divisible by 3 and by 5.

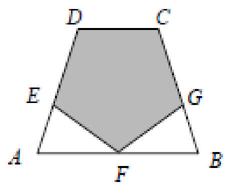
We also note that a number is divisible by 5 if, and only if, the units digit is 0 or 5.

However, our number cannot end in a 0. Indeed, in mathematics every number begins with a non-zero digit. A palindrome has equal first and last digits, so the last digit is non-zero.

Hence we seek a particular six-digit palindrome which begins and ends in 5, and which is divisible by 3.

The largest six-digit palindromes beginning and ending in 5 have the form 59dd95, for some digit d. This is divisible by 3 when the digit-sum is a multiple of 3 and therefore $5 + d$ is divisible by 3. So $d = 1$, 4, or 7. Hence the number required is 597795.

2 The diagram shows a regular pentagon $CDEFG$ inside a trapezium $ABCD$. Prove that $AB = 2 \times CD$.



Solution

First of all, we amass some basic facts.

Since the exterior angles of any polygon add up to 360°, the exterior angles of a regular pentagon are each $360° \div 5 = 72°$. Thus the interior angles are each $180° - 72° = 108°$.

Now, we claim that triangle AEF is isosceles. Indeed, $\angle AEF = 72°$ since it is an exterior angle of the pentagon. Also, since AB and DC are parallel, $\angle EAF$ is supplementary to $\angle EDC$ (they are allied angles). So $\angle EAF = 180° - 108° = 72°$ too.

This gives us that $AF = EF$.

Similarly, triangle BGF is isosceles, and $BF = GF$.

But now $AB = AF + FB = EF + FG = CD + CD$ because the sides of a regular pentagon are equal, so $AB = 2 \times CD$ as required.

3 At dinner on a camping expedition, each tin of soup was shared between 2 campers, each tin of meatballs was shared between 3 campers and each tin of chocolate pudding was shared between 4 campers. Each camper had all three courses and all tins were emptied. The camp leader opened 156 tins in total.

How many campers were on the expedition?

Solution

Each camper eats half a tin of soup, one third of a tin of meatballs, and a quarter of a tin of chocolate pudding. This is a total of

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$$

tins of food per person.

If there are N people, they thus use $\frac{13}{12}N$ tins of food in total. This gives us the equation

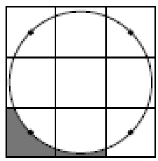
$$\frac{13}{12}N = 156,$$

so that

$$N = \frac{12}{13} \times 156$$
$$= 144.$$

Therefore there are 144 campers on the expedition.

4 The diagram shows nine 1 cm × 1 cm squares and a circle. The circle passes through the centres of the four corner squares. What is the area of the shaded region—inside two squares but outside the circle?



Solution

Consider the region of the big square which lies outside the circle. The lines in the figure divide the region into eight parts, four in the four corner squares, which are all identical by symmetry, and four in the four edge-centre squares, which are again all identical by symmetry.

The shaded area contains one part of each sort, and so takes up exactly a quarter of the difference between the big square and the circle, which we can work out.

The big square has area 9 cm^2, being made up of 9 squares each measuring $1\text{cm} \times 1\text{cm}$.

By Pythagoras' theorem, the diagonal of a $1\text{cm} \times 1\text{cm}$ square has length $\sqrt{2}$ cm. The radius of the circle is the distance from the centre of the middle square to the centre of a corner square, so is the length of one diagonal in total.

Hence the area of the circle is $\pi(\sqrt{2})^2 = 2\pi$.

Putting this all together, we see that the entire region outside the circle has area $9 - 2\pi$, and so the shaded region has area

$$\frac{9 - 2\pi}{4}.$$

5 Solve the equation $5a - ab = 9b^2$, where a and b are positive integers.

Solution

Notice that the right-hand side $9b^2$ is always positive, since the square b^2 is always positive. However, the left-hand side $5a - ab = a(5 - b)$ is only positive for $b \leqslant 4$. So, given that b is a positive integer, we can consider four cases separately, namely $b = 1, 2, 3, 4$.

If $b = 1$, then the equation becomes $(5 - 1)a = 9 \times 1^2$, that is, $4a = 9$. This has no solution for a positive integer a.

If $b = 2$, then the equation becomes $(5 - 2)a = 9 \times 2^2$, that is, $3a = 36$, so that $a = 12$. This gives the solution $a = 12$, $b = 2$.

If $b = 3$, then the equation becomes $(5 - 3)a = 9 \times 3^2$, that is, $2a = 81$. This has no solution for a positive integer a.

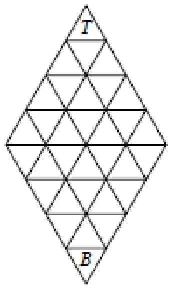
If $b = 4$, then the equation becomes $(5 - 4)a = 9 \times 4^2$, that is, $a = 144$. This gives the solution $a = 144$, $b = 4$.

Thus the solutions are $a = 12$, $b = 2$ and $a = 144$, $b = 4$.

6 A bug starts in the small triangle T at the top of the diagram. She is allowed to eat through a neighbouring edge to get to a neighbouring small triangle. So at first there is only one possible move (downwards), and only one way to reach this new triangle.

(a) How many triangles, including T and B, must the bug visit if she is to reach the small triangle B at the bottom using a route that is as short as possible?

(b) How many different ways are there for the bug to reach B from T by a route of this shortest possible length?



Solution

Shade the downwards-pointing triangles grey and leave the upwards-pointing triangles white. We separate the triangles into rows, which alternate between white and grey triangles. So, as shown in figure 1, row 1 consists of just the top triangle T, row 2 consists of the grey triangle below T, row 3 consists of the two white triangles next to it, and so on. Finally, row 14 consists of the bottom triangle B.

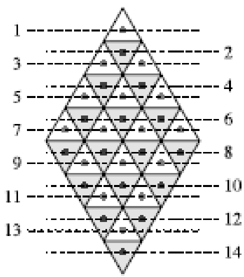


Figure 1

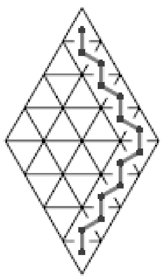


Figure 2

Notice that any move by the bug increases the row number by at most 1. So in order to get from row 1 to row 14, the bug must visit all fourteen rows on the way, so must visit at least fourteen triangles. However, the bug can do it in fourteen, for example, by following the path depicted in figure 2. This settles part (a).

For part (b), note that any path by the bug that visits only fourteen triangles must increase the row number with every step. However, from any odd-numbered row (on an upwards-pointing triangle) the only way to increase the row number is to move directly downwards.

That means we might as well join each upwards-pointing triangle with the triangle below it to form a diamond-shaped cell (as shown in figure 3): they come as a pair.

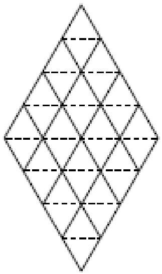


Figure 3

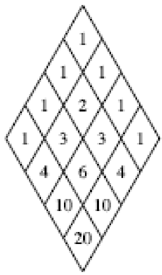


Figure 4

Using this, we can count the number of different ways of reaching each cell. Clearly there is only one way to reach the top cell.

For every cell lower down, one must first reach one of the neighbouring cells above. So the number of different ways of reaching each cell is the number of ways of reaching it from above and to the left, plus the number of ways of reaching it from above and to the right. Continuing in this way, we get the table of numbers seen in figure 4.

Hence the answer to part (b), the number of ways of reaching the bottom cell, is 20.

Alternative method for (a)

An alternative method of finding the minimum number of triangles visited is to consider the straight lines across the diagram. There are seven horizontal lines, and three lines in each diagonal direction. This makes a total of 13 straight lines. The bug wishes to cross all of them, so must cross at least 13 edges. In crossing 13 edges, the bug must visit at least 14 triangles.