Solutions to the Olympiad Cayley Paper

1 The sum of three positive integers is 11 and the sum of the cubes of these numbers is 251.

Find all such triples of numbers.

Solution

Let us calculate the first few cubes in order to see what the possibilities are:

$$1^3 = 1,\ \ 2^3 = 8,\ \ 3^3 = 27,\ \ 4^3 = 64,\ \ 5^3 = 125,\ \ 6^3 = 216 \text{ and } 7^3 = 343. \quad (*)$$

The sum of the cubes of the positive integers is 251, which is less than 343, hence none of the integers is greater than 6.

Now $\frac{251}{3} = 83\frac{2}{3} > 64 = 4^3$, therefore at least one of the integers is 5 or more.

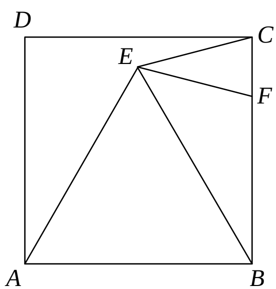
If one of the integers is 6, then the other two cubes add up to $251 - 6^3 = 251 - 216 = 35$. From (*) above, $3^3 + 2^3 = 27 + 8 = 35$ is the only possibility. Also, $6 + 3 + 2 = 11$ so that 6, 3 and 2 is a possible triple of numbers.

If one of the integers is 5, then the other two cubes add up to $251 - 5^3 = 251 - 125 = 126$. From (*) above $5^3 + 1^3 = 125 + 1 = 126$ is the only possibility. Also, $5 + 5 + 1 = 11$ so that 5, 5 and 1 is a possible triple of numbers.

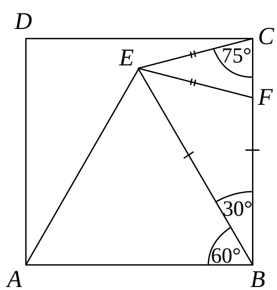
Hence 2, 3, 6 and 1, 5, 5 are the triples of numbers satisfying the given conditions.

2 The diagram shows a square *ABCD* and an equilateral triangle *ABE*. The point *F* lies on *BC* so that $EC = EF$.

Calculate the angle *BEF*.



Solution



The diagram seems to include several isosceles triangles, and we solve the problem by proving this is the case. For example, since the square *ABCD* and the equilateral triangle *ABE* share the side *AB*, all their sides are the same length. That means the triangle *BCE* is isosceles.

Now, angle *EBC* is $90° - 60° = 30°$ (since it is the difference between the interior angle of a square and the interior angle of an equilateral triangle). Hence angles *BCE* and *CEB* are each $\frac{1}{2}\left(180° - 30°\right) = 75°$, because they are the base angles of an isosceles triangle.

We are also given that triangle *CEF* is isosceles. Since we have worked out that angle $FCE = 75°$, we deduce that angle $CEF = 180° - \left(2 \times 75°\right) = 30°$.

Finally, we find that angle $BEF = \angle CEB - \angle CEF = 75° - 30° = 45°$.

3 Find all possible solutions to the 'word sum' on the right.

Each letter stands for one of the digits 0−9 and has the same meaning each time it occurs. Different letters stand for different digits. No number starts with a zero.

$$\begin{array}{r} O\ D\ D \\ +\ O\ D\ D \\ \hline E\ V\ E\ N \end{array}$$

Solution

Firstly, it is clear that the three-digit number 'ODD' lies between 100 and 999. Therefore, since 'EVEN' = 2 × 'ODD', we have

$$200 < \text{'EVEN'} < 1998.$$

Hence the first digit E of 'EVEN' is 1 since it is a four-digit number.

We are left with the following problem:

$$\begin{array}{r} O\ D\ D \\ +\ O\ D\ D \\ \hline 1\ V\ 1\ N \\ \hline \end{array}$$

Now the same numbers are added in the tens and units columns, but N ≠ 1, otherwise N and E would be equal. The only way for different totals to occur in these columns is for there to be a 'carry' to the tens column, and the greatest possible carry is 1, so that N = 0.

There are two possible digits D that give N = 0, namely 0 and 5. But 0 is already taken as the value of N, so that D = 5. The problem is thus:

$$\begin{array}{r} O\ 5\ 5 \\ +\ O\ 5\ 5 \\ \hline 1\ V\ 1\ 0 \\ \hline \end{array}$$

Now, the digit O has to be big enough to produce a carry, but cannot be 5, which is already taken as the value of D. So the possibilities are

$$\begin{array}{r} 6\ 5\ 5 \\ +\ 6\ 5\ 5 \\ \hline 1\ 3\ 1\ 0 \\ \hline \end{array} \qquad \begin{array}{r} 7\ 5\ 5 \\ +\ 7\ 5\ 5 \\ \hline 1\ 5\ 1\ 0 \\ \hline \end{array} \qquad \begin{array}{r} 8\ 5\ 5 \\ +\ 8\ 5\ 5 \\ \hline 1\ 7\ 1\ 0 \\ \hline \end{array} \qquad \begin{array}{r} 9\ 5\ 5 \\ +\ 9\ 5\ 5 \\ \hline 1\ 9\ 1\ 0 \\ \hline \end{array}$$

but the second and fourth of these are not allowed since V repeats a digit used for another letter. We are left with the two possibilities

$$\begin{array}{r} 6\ 5\ 5 \\ +\ 6\ 5\ 5 \\ \hline 1\ 3\ 1\ 0 \\ \hline \end{array} \qquad \begin{array}{r} 8\ 5\ 5 \\ +\ 8\ 5\ 5 \\ \hline 1\ 7\ 1\ 0 \\ \hline \end{array}$$

and it is clear that both of these work.

4 Walking at constant speeds, Eoin and his sister Angharad take 40 minutes and 60 minutes respectively to walk to the nearest town.

Yesterday, Eoin left home 12 minutes after Angharad. How long was it before he caught up with her?

Solution

Let the distance from home to town be D km. Now in every minute Eoin travels one-fortieth of the way to town: that is, a distance of $\frac{D}{40}$ km. So after t minutes, he has travelled a distance

$$\frac{tD}{40}\ \text{km}.$$

Similarly, in every minute Angharad travels one-sixtieth of the way to town: that is, a distance of $\frac{D}{60}$ km. But she has had 12 minutes extra walking time. So after Eoin has been walking for t minutes, she has been walking for $t + 12$ minutes and so has travelled a distance

$$\frac{(t + 12)D}{60}\ \text{km}.$$

We are asked how long Eoin has been walking when they meet. They meet when they have travelled equal distances, which is when

$$\frac{tD}{40} = \frac{(t + 12)D}{60}.$$

We cancel the D from each side and multiply both sides by 120 to obtain

$$120 \times \frac{t}{40} = 120 \times \frac{t + 12}{60}.$$

Simplifying, we get

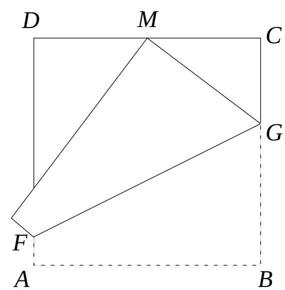
$$3t = 2(t + 12),$$

which we solve to give $t = 24$.

Thus Eoin catches up with Angharad after he has walked for 24 minutes.

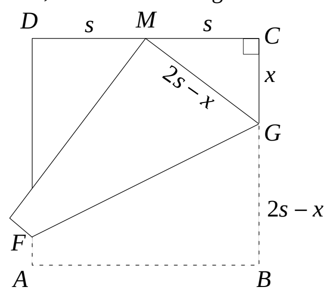
5 A square sheet of paper $ABCD$ is folded along FG, as shown, so that the corner B is folded onto the midpoint M of CD.

Prove that the sides of triangle GCM have lengths in the ratio 3 : 4 : 5.



Solution

This problem does not give us units, and so we choose them so that the side length of the square is $2s$. Since M is the midpoint of CD, we have $CM = s$. Then we define $x = CG$. Since $BC = 2s$, $GB = 2s - x$. But, as GM is the image of GB after folding, $GM = 2s - x$ too.



Now Pythagoras' theorem for triangle MCG gives us

$$s^2 + x^2 = (2s - x)^2.$$

We multiply out to get

$$s^2 + x^2 = 4s^2 - 4sx + x^2.$$

Eliminating the x^2 terms and dividing by s (which is not zero), we obtain

$$s = 4s - 4x,$$

which has the solution $x = \frac{3}{4}s$.

Thus the triangle GCM has sides of length $x = \frac{3}{4}s$, s and $2s - x = \frac{5}{4}s$. Multiplying all the sides by 4, we get $3s$, $4s$ and $5s$, so the side lengths are in the ratio $3 : 4 : 5$, as required.

6 A 'qprime' number is a positive integer which is the product of exactly two different primes, that is, one of the form $q \times p$, where q and p are prime and $q \neq p$.

What is the length of the longest possible sequence of *consecutive* integers all of which are qprime numbers?

Solution

To help to understand this problem, it is natural to test the first few numbers to see which small numbers are qprime, and which are not:

1 is not a qprime since it has no prime factors.

2 and 3 are not qprimes since they are prime.

4 is not qprime since it is 2×2.

5 is not qprime, since it is prime.

$6 = 2 \times 3$ is the first qprime number.

7 is not qprime.

8 is not qprime, since it is $2 \times 2 \times 2$.

9 is not, since it is 3×3.

$10 = 2 \times 5$ is another qprime.

11 is not.

12 is not, since it is $2 \times 2 \times 3$.

Of course, we cannot prove a general result just by continuing the list, but it can guide us to a proof, such as the one that follows.

We note that no multiple of 4 is ever qprime, since a multiple of 4 is a multiple of 2×2. This means that a string of consecutive qprime numbers can be of length at most three, because any sequence of four or more consecutive integers includes a multiple of 4.

We are therefore led to ask whether any strings of three consecutive qprime numbers exist. We have looked as far as 12 and not found any, but we will continue searching, using the fact that none of the numbers is a multiple of 4:

For (13, 14, 15), the number 13 is prime and so not qprime.

For (17, 18, 19), 17 is not qprime (nor are the others).

For (21, 22, 23), 23 is not qprime.

For (25, 26, 27), 25 is not qprime (nor is 27).

For (29, 30, 31), 29 is not qprime (nor are the others).

For (33, 34, 35), all three are qprime (being 3×11, 2×17 and 5×7).

So we have found a sequence of three consecutive qprimes, and have also proved that no sequence of four (or more) consecutive qprimes exists.

Thus the longest possible sequence of consecutive integers all of which are qprime numbers has length 3.