Solutions to the Olympiad Cayley Paper

1. An aquarium contains 280 tropical fish of various kinds. If 60 more clownfish were added to the aquarium, the proportion of clownfish would be doubled.
How many clownfish are in the aquarium?

Solution

Let there be x clownfish in the aquarium.

If 60 clownfish are added there are $x + 60$ clownfish and 340 tropical fish in total.

Since the proportion of clownfish is then doubled, we have

$$2 \times \frac{x}{280} = \frac{x + 60}{340}.$$

Multiplying both sides by 20 we get

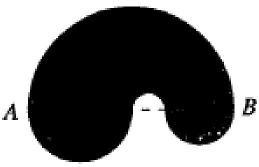
$$\frac{x}{7} = \frac{x + 60}{17}$$

and hence

$$17x = 7(x + 60).$$

It follows that $x = 42$ and thus there are 42 clownfish in the aquarium.

2. The boundary of the shaded figure consists of four semicircular arcs whose radii are all different. The centre of each arc lies on the line AB, which is 10 cm long.
What is the length of the perimeter of the figure?



Solution

The centre of the large semicircular arc lies on AB, so we know that AB is a diameter of the large semicircle. But AB is 10 cm long, so the radius of the large semicircle is 5 cm.

Let the radii of the other three semicircles be r_1 cm, r_2 cm and r_3 cm. The centres of these arcs also lie on AB, so the sum of their diameters is equal to the length of AB. It follows that $2r_1 + 2r_2 + 2r_3 = 10$ and hence $r_1 + r_2 + r_3 = 5$.

Now the lengths, in cm, of the semicircular arcs are 5π, πr_1, πr_2 and πr_3. Therefore the perimeter of the figure has length, in cm,

$$\begin{aligned} 5\pi + \pi r_1 + \pi r_2 + \pi r_3 &= \pi(5 + r_1 + r_2 + r_3) \\ &= \pi(5 + 5) \\ &= 10\pi. \end{aligned}$$

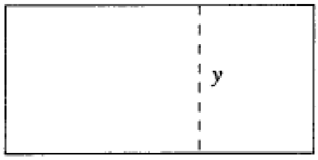
Hence the perimeter of the figure has length 10π cm.

3. Two different rectangles are placed together, edge-to-edge, to form a large rectangle. The length of the perimeter of the large rectangle is $\frac{2}{3}$ of the total perimeter of the original two rectangles.

   Prove that the final rectangle is in fact a square.

First solution

Since the smaller rectangles are placed together edge-to-edge, they have a side length in common. Let this side have length y and let the other sides have lengths x_1 and x_2 as shown.



x_1 x_2

The perimeters of the smaller rectangles are $2x_1 + 2y$ and $2x_2 + 2y$, so the total perimeter of the two smaller rectangles is $2x_1 + 2x_2 + 4y$.

The perimeter of the large rectangle is $2(x_1 + x_2) + 2y = 2x_1 + 2x_2 + 2y$.

We are given that the length of the perimeter of the large rectangle is $\frac{2}{3}$ of the total perimeter of the two original rectangles. Hence we may form the equation

$$2x_1 + 2x_2 + 2y = \tfrac{2}{3}(2x_1 + 2x_2 + 4y).$$

We may simplify this equation by multiplying both sides by 3 and expanding the brackets, to obtain

$$6x_1 + 6x_2 + 6y = 4x_1 + 4x_2 + 8y,$$

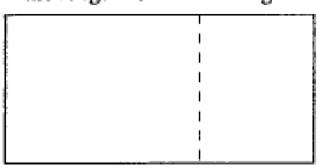
which simplifies to

$$x_1 + x_2 = y.$$

This means that the length and width of the large rectangle are the same. In other words, the rectangle is actually a square.

Second solution

The total perimeter length P of the original two rectangles is equal to the perimeter length of the large rectangle added to the lengths of the two edges which are joined together.



But the perimeter length of the large rectangle is $\frac{2}{3}P$ and hence the two edges which are joined together have total length $\frac{1}{3}P$.

However, the two edges which are joined together are parallel to two sides of the large rectangle and have the same length as them. Hence these two sides of the large rectangle have total length $\frac{1}{3}P$.

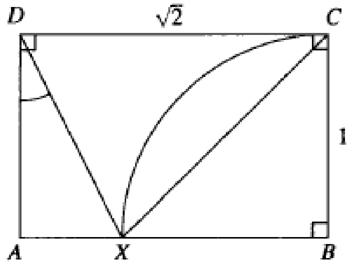
Since the perimeter length of the large rectangle is $\frac{2}{3}P$, the other two sides of the large rectangle also have total length $\frac{1}{3}P$. It follows that all the sides of the rectangle are equal in length, in other words, the rectangle is a square.

4. In the rectangle *ABCD*, the side *AB* has length $\sqrt{2}$ and the side *AD* has length 1. Let the circle with centre *B* and passing through *C* meet *AB* at *X*.

Find $\angle ADX$ (in degrees).

Solution

We begin with a diagram showing the information given in the question. We have used the fact that *ABCD* is a rectangle, so that $BC = AD = 1$ and $DC = AB = \sqrt{2}$, and angles *ABC*, *BCD* and *CDA* are right angles.



Since *BX* and *BC* are both radii of the circle, *BX* also has length 1. This means that triangle *XBC* is isosceles and so $\angle BXC = \angle BCX$.

Furthermore, since $\angle ABC$ is a right angle, $\angle BXC$ and $\angle BCX$ are both equal to 45°.

From the fact that $\angle BCD$ is a right angle, it follows that $\angle XCD = 90° - 45° = 45°$.

We may use Pythagoras' theorem in triangle *XBC* to obtain

$$\begin{aligned} XC^2 &= BX^2 + BC^2 \\ &= 1^2 + 1^2 \\ &= 2 \end{aligned}$$

and so $XC = \sqrt{2}$.

We are given that *DC* also has length $\sqrt{2}$ and so triangle *XCD* is isosceles. This means that $\angle CXD$ and $\angle CDX$ are equal, and so each is equal to $(180° - 45°) \div 2 = 67\tfrac{1}{2}°$.

Lastly, we use the fact that $\angle CDA$ is a right angle to conclude that $\angle ADX = 90° - 67\tfrac{1}{2}° = 22\tfrac{1}{2}°$.

5. Two candles are the same height. The first takes 10 hours to burn completely whilst the second takes 8 hours to burn completely.

   Both candles are lit at midday. At what time is the height of the first candle twice the height of the second candle?

   Solution

   Let the initial height of each candle be h cm. In one hour the first candle will burn $\frac{h}{10}$ cm and the second candle will burn $\frac{h}{8}$ cm. Thus in t hours, the candles will burn

$$\frac{ht}{10}\text{ cm} \qquad \text{and} \qquad \frac{ht}{8}\text{ cm,}$$

   respectively.

   If both candles are lit at midday, then t hours after midday the heights of the first and second candles will be

$$\left(h - \frac{ht}{10}\right)\text{ cm} \qquad \text{and} \qquad \left(h - \frac{ht}{8}\right)\text{ cm,}$$

   respectively.

   We are asked to find the time at which the height of the first candle is twice the height of the second candle. We therefore need to find the value of t such that

$$h - \frac{ht}{10} = 2\left(h - \frac{ht}{8}\right).$$

   We may divide every term by h, since we know that h is not zero, and expand the brackets to obtain the equation

$$1 - \frac{t}{10} = 2 - \frac{t}{4}.$$

   Multiplying both sides by 20, we get

$$20 - 2t = 40 - 5t,$$

   and so

$$t = \frac{20}{3}$$

$$= 6\tfrac{2}{3}.$$

   Hence the height of the first candle is twice that of the second after 6 hours and 40 minutes, in other words, this happens at 18:40.

6. Teams A, B, C and D competed against each other once. The results table was as follows:

Team	Win	Draw	Loss	Goals for	Goals against
A	3	0	0	5	1
B	1	1	1	2	2
C	0	2	1	5	6
D	0	1	2	3	6

(a) Find (with proof) which team won in each of the six matches.

(b) Find (with proof) the scores in each of the six matches.

Solution

(a) Team A won all three games and so beat teams B, C and D.

Of the three games that team C played, the one that was lost can only have been against team A. Therefore team C drew against teams B and D.

If we consider the three games that team B played, the game against team A was lost, the game against team C was a draw and so the remaining game, that team B won, was against team D.

In summary:

A beat B, A beat C, A beat D;

B drew with C, B beat D; and

C drew with D.

(b) Consider the following table in which the rows give the number of goals scored *for* each team and the columns give the number of goals *against* each team.

		Goals against				
		A	B	C	D	All
Goals for	A	–		$z + 1$		5
	B		–	x	t	2
	C	z	x	–	y	5
	D			y	–	3
	All	1	2	6	6	15

We have let the number of goals scored by team C against team B be x, so that the number of goals scored by team B against team C is also x, since their match was a draw. Similarly, we have let the number of goals scored by team C against team D be y, so that this is also the number scored by team D against team C.

Furthermore, we have let the number of goals scored by team C against team A be z, so that the number of goals scored by team A against team C is $z + 1$ since the difference between the number of goals scored and conceded by team C is 1.

Finally, we have let the number of goals scored by team B against team D be t. Then t is at least 1 since team B beat team D.

We observe that the row for C now means that $x + y + z = 5$ (which agrees with the column for C).

From the column for A we see that z is at most 1, since the total in that column is 1. Similarly, from the row for D, we see that y is at most 3, and from the row for B we see that x is at most 1 since t is at least 1.

But we have $x + y + z = 5$, so that the only possibilities are $x = 1, y = 3$ and $z = 1$. It follows that $t = 1$.

Therefore the table is:

Goals against

		A	B	C	D	All
	A	–		2		5
Goals	B		–	1	1	2
for	C	1	1	–	3	5
	D			3	–	3
	All	1	2	6	6	15

We may now complete the table by, for example, first noting that all other entries in the column for A are 0, and then filling in the rows from the bottom.

Goals against

		A	B	C	D	All
	A	–	1	2	2	5
Goals	B	0	–	1	1	2
for	C	1	1	–	3	5
	D	0	0	3	–	3
	All	1	2	6	6	15

In summary, the scores in each match were as follows:

A beat B	1 – 0
A beat C	2 – 1
A beat D	2 – 0
B drew with C	1 – 1
B beat D	1 – 0
C drew with D	3 – 3