Solutions to the Olympiad Cayley Paper

1. How many four-digit multiples of 9 consist of four different odd digits?

First solution

There are five odd digits: 1, 3, 5, 7 and 9.

The sum of the four smallest odd digits is 16 and the sum of the four largest is 24. Hence the digit sum of any four-digit number with different odd digits lies between 16 and 24, inclusive.

However, the sum of the digits of a multiple of 9 is also a multiple of 9, and the only multiple of 9 between 16 and 24 is 18. Hence the sum of the four digits is 18.

Now $1 + 3 + 5 + 9 = 18$, so that the four digits *can* be 1, 3, 5 and 9. If 7 is one of the four digits then the sum of the other three is 11, which is impossible. So 7 cannot be one of the digits and therefore the four digits can only be 1, 3, 5 and 9.

The number of arrangements of these four digits is $4 \times 3 \times 2 \times 1 = 24$. Hence there are 24 four-digit multiples of 9 that consist of four different odd digits.

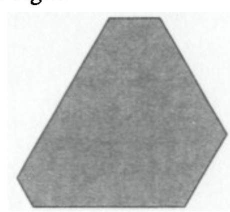
Second solution

The sum of all five odd digits is $1 + 3 + 5 + 7 + 9 = 25$.
Subtracting 1, 3, 5, 7 and 9 in turn we get 24, 22, 20, 18 and 16, only one of which is a multiple of 9, namely $18 = 25 - 7$. Since the sum of the digits of a multiple of 9 is also a multiple of 9, it follows that the four digits can only be 1, 3, 5 and 9.

The number of arrangements of these four digits is $4 \times 3 \times 2 \times 1 = 24$. Hence there are 24 four-digit multiples of 9 that consist of four different odd digits.

2. A hexagon is made by cutting a small equilateral triangle from each corner of a larger equilateral triangle. The sides of the smaller triangles have lengths 1, 2 and 3 units. The lengths of the perimeters of the hexagon and the original triangle are in the ratio 5 : 7.

What fraction of the area of the original triangle remains?



First solution

Let the side length of the large equilateral triangle be x units; this triangle therefore has a perimeter of length $3x$ units.

Now consider the hexagon, which has sides of lengths $1, x - 3,\ 2, x - 5,\ 3$ and $x - 4$ units. Hence the hexagon has perimeter length $3x - 6$ units.

Since the ratio of the perimeter lengths of the hexagon and the large triangle is 5 : 7, we have

$$\frac{3x - 6}{3x} = \frac{5}{7}.$$

Rearranging and solving for x we obtain

$$x = 7. \qquad (*)$$

Now, in order to find the area of the large equilateral triangle, we determine the height h units using Pythagoras' theorem:

$$h^2 = 7^2 - (\tfrac{7}{2})^2$$

$$= 49(1 - \tfrac{1}{4}) = 49 \times \tfrac{3}{4}.$$

Hence $$h = \sqrt{49 \times \frac{3}{4}} = 7 \times \frac{\sqrt{3}}{2}.$$

Therefore the area of the large equilateral triangle is

$$\frac{1}{2} \times 7 \times \frac{7\sqrt{3}}{2} = \frac{49\sqrt{3}}{4}.$$

We may find the areas of the three small equilateral triangles in a similar way. These areas are

$$\frac{\sqrt{3}}{4}, \frac{4\sqrt{3}}{4} \text{ and } \frac{9\sqrt{3}}{4}.$$

The area of the hexagon is the area of the large equilateral triangle minus the areas of the three small equilateral triangles, that is,

$$\frac{49\sqrt{3}}{4} - \left(\frac{\sqrt{3}}{4} + \frac{4\sqrt{3}}{4} + \frac{9\sqrt{3}}{4}\right) = \frac{35\sqrt{3}}{4}.$$

Finally, the fraction of the original equilateral triangle remaining is

$$\frac{35\sqrt{3}}{4} \div \frac{49\sqrt{3}}{4} = \frac{5}{7}.$$

Second solution

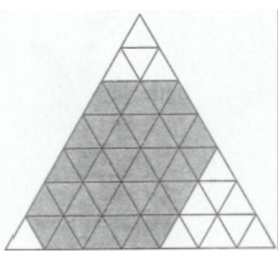
Having established that the large triangle has sides of length 7 (equation (*) in the solution above), we may proceed as follows:

The four equilateral triangles in the problem are similar. Now the ratio of the areas of similar figures is equal to the ratio of the squares of their sides. Hence the four triangles have areas in the ratio $1^2 : 2^2 : 3^2 : 7^2 = 1 : 4 : 9 : 49$.

Hence the ratio of the areas of the hexagon and the large triangle is

$$49 - (1 + 4 + 9) : 49 = 35 : 49 = 5 : 7.$$

This may be illustrated by dividing the large triangle into 49 small triangles, as shown.



Note:

The observant reader will have noticed that the answer to this problem is surprising: the ratio of the areas is the same as the ratio of the perimeters. There is no reason to expect this to happen.

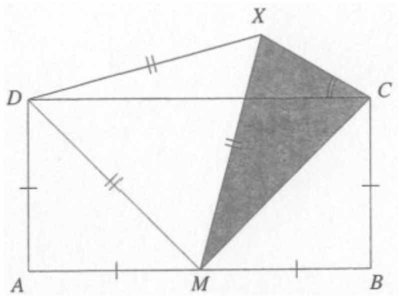
3. In the rectangle $ABCD$ the midpoint of AB is M and $AB : AD = 2 : 1$. The point X is such that triangle MDX is equilateral, with X and A lying on opposite sides of the line MD.

Find the value of $\angle XCD$.

Solution

The key to this solution is to draw MC and consider triangle MCX.

We are given that $ABCD$ is a rectangle, so that $BC = AD$ and $\angle DAM = 90° = \angle MBC$.
We are also given that $AB = 2AD$ and that M is the midpoint of AB. Therefore $DA = AM = MB = BC$.



It follows that triangles DAM and MBC are congruent (SAS) and we deduce that $DM = MC$.

But triangle MDX is equilateral, so $MX = DM$ and hence $MX = MC$. In other words, triangle MCX is isosceles.

Now consider the angles at M.

1. Triangle DAM is right-angled with $\angle DAM = 90°$. It is also isosceles, so $\angle AMD = \angle ADM = 45°$, since the angle sum is 180°.
2. Similarly, from triangle MBC, $\angle CMB = 45°$.
3. Finally, because triangle MDX is equilateral, $\angle DMX = 60°$.

Hence
$$\begin{aligned}\angle CMX &= 180° - 45° - 45° - 60° \\ &= 30°\end{aligned}$$
since angles on a straight line add up to 180°.

Lastly, we consider the angles at C.

We know that triangle XCM is isosceles and that $\angle CMX = 30°$. Hence each base angle is $\frac{1}{2}(180° - 30°) = 75°$; in particular, $\angle XCM = 75°$.

Also, $ABCD$ is a rectangle, so $\angle DCB = 90°$, and triangle MBC is right-angled and isosceles, so $\angle MCB = 45°$. Therefore
$$\begin{aligned}\angle DCM &= \angle DCB - \angle MCB \\ &= 90° - 45° \\ &= 45°.\end{aligned}$$

We can now calculate the value of $\angle XCD$.

We have
$$\begin{aligned}\angle XCD &= \angle XCM - \angle DCM \\ &= 75° - 45° \\ &= 30°.\end{aligned}$$

4. The number N is the product of the first 99 positive integers. The number M is the product of the first 99 positive integers after each has been reversed. That is, for example, the reverse of 8 is 8; of 17 is 71; and of 20 is 02.

Find the exact value of $N \div M$.

First solution

From the given definition we have

$$N = (1 \times 2 \times \ldots \times 9) \times 10 \times (11 \times 12 \times \ldots \times 19) \times 20 \times \ldots \times 90 \times (91 \times \ldots \times 99),$$

which rearranges to

$$N = (1 \times 2 \times \ldots \times 9) \times (11 \times 12 \times \ldots \times 19) \times \ldots \times (91 \times \ldots \times 99) \times (10 \times 20 \times \ldots \times 90).$$

Also

$$M = (1 \times 2 \times \ldots \times 9) \times 01 \times (11 \times 21 \times \ldots \times 91) \times 02 \times \ldots \times 09 \times (19 \times \ldots \times 99)$$

which rearranges to

$$M = (1 \times 2 \times \ldots \times 9) \times (11 \times 12 \times \ldots \times 19) \times \ldots \times (91 \times \ldots \times 99) \times (01 \times 02 \times \ldots \times 09)$$

$$= (1 \times 2 \times \ldots \times 9) \times (11 \times 12 \times \ldots \times 19) \times \ldots \times (91 \times \ldots \times 99) \times (1 \times 2 \times \ldots \times 9).$$

Comparing these arrangements for M and N, we see that M has the same terms as N except that the product $10 \times 20 \times \ldots \times 90$ is replaced by the product $1 \times 2 \times \ldots \times 9$. Thus when we divide N by M all the common terms cancel and we are left with

$$\frac{N}{M} = \frac{10 \times 20 \times \ldots \times 90}{1 \times 2 \times \ldots \times 9}$$

$$= 10^9.$$

Second solution

We may place the numbers from 1 to 99 into three categories, determined by how they are transformed when they are reversed:

1. single digit numbers 'a' are unchanged;
2. a two-digit number 'ab', where neither a nor b is zero, is transformed to the two-digit number 'ba'; and
3. a multiple of 10 such as 'a0' is transformed to '0a' $=$ a, a single-digit number.

Thus there is a correspondence between the factors in N and M, as shown in the table:

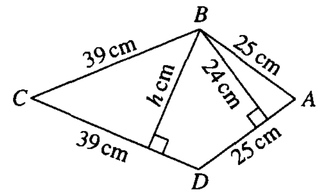
N	M
'a'	'a'
'aa'	'aa'
'ab' and 'ba'	'ba' and 'ab'
'a0'	'a'

Single-digit numbers are unchanged; two-digit numbers with a repeated digit are unchanged; pairs of two-digit numbers, with different digits and neither digit zero, are unchanged as a pair; the multiples of 10 in N are replaced by single-digit numbers in M. Thus when we divide N by M all the identical factors cancel and we are left with

$$\frac{N}{M} = \frac{10 \times 20 \times \ldots \times 90}{1 \times 2 \times \ldots \times 9}$$

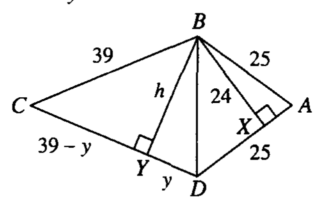
$$= 10^9.$$

5. A kite has sides AB and AD of length 25 cm and sides CB and CD of length 39 cm. The perpendicular distance from B to AD is 24 cm. The perpendicular distance from B to CD is h cm.
Find the value of h.



First solution

As shown in the figure below, let the perpendicular from B to the line AD meet the line AD at the point X; let the perpendicular from B to the line CD meet the line CD at the point Y and let the distance DY be y cm.



Considering triangle ABX and using Pythagoras' Theorem we obtain

$$AX = \sqrt{25^2 - 24^2} \text{ cm}$$
$$= 7 \text{ cm.}$$

Similarly, from triangle BDX we have

$$BD = \sqrt{24^2 + (25 - 7)^2} \text{ cm}$$
$$= 30 \text{ cm.} \qquad (*)$$

Now from triangles BDY and BCY, again by Pythagoras' theorem, we deduce that

$$h^2 + (39 - y)^2 = 39^2$$

and $$h^2 + y^2 = 30^2. \qquad (1)$$

Subtract to get $$(39 - y)^2 - y^2 = 39^2 - 30^2,$$

which simplifies to $$78y = 900,$$

so that $$y = \frac{150}{13}.$$

Finally, by substituting in equation (1), we find

$$h = \frac{360}{13}.$$

Second solution

Another solution uses the length of BD obtained in (*) above to find the area of isosceles triangle BCD. Once the area is known the value of the height h may be found from

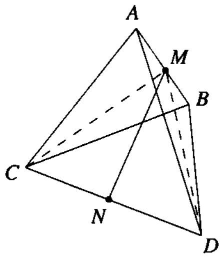
$$\text{area} = \tfrac{1}{2} \times 39 \times h.$$

Can you see how to find the area of triangle BCD and so complete the solution?

Note: Triangle DYB is a '5, 12, 13' triangle.

6. A regular tetrahedron $ABCD$ has edges of length 2 units. The midpoint of the edge AB is M and the midpoint of the edge CD is N.

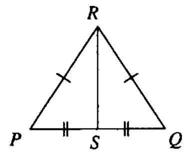
Find the exact length of the segment MN.



First solution

We make use of the following result.

Theorem (Median of isosceles triangle): The line joining the apex to the midpoint of the base of an isosceles triangle is perpendicular to the base. That is, in the following figure, $\angle PSR = 90°$.



Applying the theorem to triangle ABC, we find that $\angle AMC = 90°$. Similarly, in triangle ABD, $\angle AMD = 90°$.

Now applying Pythagoras' theorem to the triangles AMC and AMD we get

$$CM^2 = AC^2 - AM^2 = 2^2 - 1^2$$

and $$DM^2 = AD^2 - AM^2 = 2^2 - 1^2.$$

Hence $CM = \sqrt{3}$ and $DM = \sqrt{3}$, so triangle CMD is isosceles.

Now apply the theorem to triangle CMD to obtain $\angle CNM = 90°$.

Then by Pythagoras' theorem in triangle CNM

$$\begin{aligned} MN^2 &= CM^2 - CN^2 \\ &= 3 - 1. \end{aligned}$$

Therefore $MN = \sqrt{2}$.

Second solution

A tetrahedron may be formed by joining face diagonals of a cube, as shown below.

Since the faces of the cube are congruent squares the face diagonals have equal length and so the tetrahedron is regular.

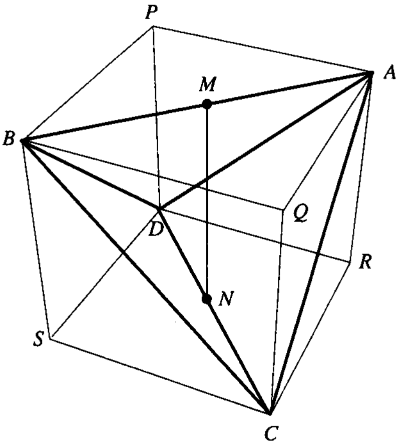
Now M and N are midpoints of opposite edges of the tetrahedron. Therefore they are midpoints of opposite face diagonals of the cube, that is, centres of opposite faces of the cube. Hence $MN = AR$.

Letting the sides of the cube have length a, from Pythagoras' theorem in triangle ARC we get

$$AC^2 = AR^2 + RC^2$$

so that $$\begin{aligned} 2^2 &= a^2 + a^2 \\ &= 2a^2. \end{aligned}$$

Hence $a = \sqrt{2}$ and therefore $MN = \sqrt{2}$.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