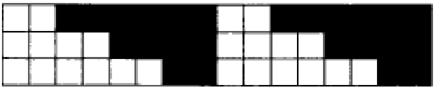
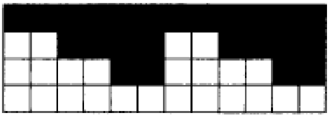
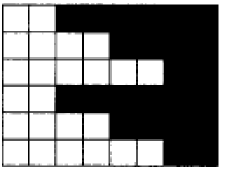
Solutions to the Olympiad Cayley Paper

1. The area of the polygon is 12 square units so that fitting together four copies of the polygon will create a rectangle with an area of 48 square units. The only possible integer dimensions for such a rectangle are 1×48, 2×24, 3×16, 4×12 and 6×8. Since the polygon has a height of 3 units and a length of 6 units, the first two rectangles are impossible. The diagrams show that the other three are possible. Hence three different rectangles can be made.







2. Let x be the sum the marks scored in the tests so far and let n be the total number of tests. From the given information we have $\frac{x + 17}{n} = 80$ and $\frac{x + 92}{n} = 85$. Rearranging each of these gives $x + 17 = 80n$ and $x + 92 = 85n$, and subtracting these equations gives $75 = 5n$. Hence the total number, n, of tests is 15.

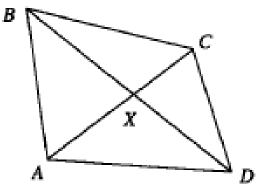
3. The square is divided into five equal areas, so each is equal to 20 square units. Considering the area of triangle ADP gives $20 = \frac{1}{2} \times 10 \times DP$. Hence $DP = 4$ units. Similarly, $BS = 4$ units, $PQ = 4$ units and $SR = 4$ units. Hence, $QC = CR = 2$ units. Using Pythagoras' theorem, $QR = \sqrt{QC^2 + CR^2} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$ units.

4. Consider the possible right angles at a vertex. A right angle may be formed by two edges (in 3 ways), or by an edge and a face diagonal (in 3 ways). This is the same for each of the eight vertices. Hence the total number of triangles is $8 \times 6 = 48$.

   Alternatively:

   Consider all possible rectangles whose vertices are vertices of the cube. There are 6 faces and 6 diagonal rectangles. Each diagonal of such a rectangle determines two right-angled triangles, so each rectangle corresponds to 4 right-angled triangles. Hence the total number of triangles is $12 \times 4 = 48$.

5. We are given that $\angle BXC = 100°$, so that $\angle AXD$ is also $100°$ as it is vertically opposite. Considering the sum of the angles in triangle ADX, we deduce that $\angle BDA = 40°$; considering the sum of the angles of triangle ABD, we deduce that $\angle DBA = 40°$. Hence $\angle BDA = \angle DBA$, so that $AD = AB$. Now in triangle ABC we have $AB = BC$ and $\angle BAC = 60°$; therefore triangle ABC is equilateral and $AC = AB$. It follows that $AC = AD$, so that triangle ACD is isosceles and hence $\angle ADC = \angle ACD = 70°$, since $\angle CAD = 40°$. Therefore $\angle BDC = \angle ADC - \angle BDA = 70° - 40° = 30°$.



6. (a) Let the number of integers in the list be n. Then considering the total of all the numbers gives $47n = 329$ and therefore $n = 7$. Let X be the largest possible value for a number in the list. Since the list has a fixed sum, the maximum value of X occurs when the other numbers are as small as possible, that is, when the list is 1, 2, 3, 4, 5, 97, X (remembering that the numbers are distinct, positive integers). But the total is 329, so $X = 217$.

(b) Let Y be the largest number in the list of positive integers. We wish to maximise Y and yet maintain an average of 47.

Form a new list of integers by subtracting 47 from every number in the original list. Then the average of the numbers in the new list is zero and therefore (the sum of the positive numbers) + (the sum of the negative numbers) = 0.

Since $Y - 47$ is one of the positive numbers in the new list, to maximise Y the new list needs to have as many negative numbers as possible (thereby making the sum of the positive numbers as large as possible), but as few positive numbers as possible (thereby making Y as large as possible).

In other words, the original list needs to have as many integers as possible smaller than 47 and as few integers as possible greater than 47. One such list of numbers is 1, 2, 3, 4,…, 45, 46, 97, Y. Since there are 48 integers in this list and the average is 47, their sum is $48 \times 47 = 2256$ and hence Y is 1078. (Note that other such lists are possible, which include the same numbers as the one considered but also include 47 any number of times. These all give the same value for Y.)