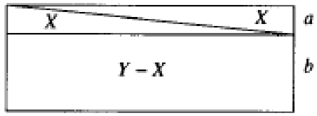
Solutions to the Olympiad Cayley Paper

1. Let the width of the rectangle be l. Now region X is a triangle and region Y is a trapezium, so that area $X = \frac{1}{2}al$ and area $Y = \frac{1}{2}(a + 2b)l$. Since area X : area $Y = 2 : 7$, we have $\dfrac{\frac{1}{2}al}{\frac{1}{2}(a + 2b)l} = \dfrac{2}{7}$. Rearranging and simplifying gives $5a = 4b$. Hence $\dfrac{a}{b} = \dfrac{4}{5}$ so the ratio $a : b$ equals 4 : 5.

Alternatively

Since area X : area $Y = 2 : 7$ there is a positive number k such that area $X = 2k$ and area $Y = 7k$.



Divide area Y into two parts, with areas X and $(Y - X)$, by drawing a line parallel to the base of the rectangle, as shown. The original rectangle has now been divided into two rectangles with equal width and with areas $2X$ and $Y - X$. Hence $a : b = 2X : (Y - X)$. Substituting for X and Y gives $a : b = 4k : (7k - 2k) = 4 : 5$.

2. In the "letter sum", label the units column **1**, the tens column **2**, and so on, as shown.

	5	4	3	2	1
	S	E	V	E	N
+			O	N	E
	E	I	G	H	T

In column **5**, since different letters represent different digits, then S cannot be equal to E. This shows that a '1' is carried from column **4** to column **5**. Therefore E = 9 and S = 8. Rewriting the sum with these values gives:

	5	4	3	2	1
	8	9	V	9	N
+			O	N	9
	9	I	G	H	T

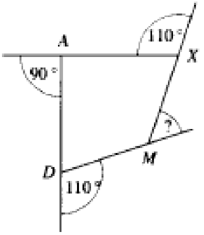
Consider column **1**. If N = 0 then T = 9 and so T = E, which contradicts the statement that different letters represent different digits. Hence N is non-zero and a '1' is carried from column **1** to column **2**. Comparing columns **1** and **2** shows that H = T + 1, since the columns are identical except for the '1' carried to column **2**. However, column **1** shows that N + 9 = T + 10 so that T = N − 1. Hence H = (N − 1) + 1 = N, which contradicts the statement that different letters represent different numbers. Therefore there are no solutions to this "letter sum".

3. Since rectangles $ABCD$ and $AZYX$ are congruent, $\angle AXZ = \angle ADB = 70°$, and from right-angled triangle ABD we have $\angle ABD = 20°$. Now angle AXZ is an exterior angle of triangle BMX, so that $\angle AXZ = \angle ABD + \angle BMX$. Hence $70° = 20° + \angle BMX$ and therefore $\angle BMX = 50°$.

Alternatively

Since rectangles $ABCD$ and $AZYX$ are congruent, $\angle AXZ = \angle ADB = 70°$. Using angles on a straight line, we may deduce three exterior angles of quadrilateral $ADMX$, as shown.

But the sum of the exterior angles of a polygon is 360°, so that $\angle BMX = 50°$.



4. For a positive integer with fewer than four digits, adding a 5 at each end increases the number by less than 59995. Similarly, a number with more than four digits increases by more than 5000005. Since the required number is increased by 518059, it therefore has four digits. Let this four digit integer be '$DCBA$' where each letter stands for a digit from 0 to 9, except that D is not zero. We can write the given information as a sum:

	5	1	8	0	5	9
+			D	C	B	A
	5	D	C	B	A	5

From the units column, $A = 6$. Continuing to work from right to left gives $B = 0$, $C = 0$ and $D = 2$. Hence the value of '$DCBA$' is 2006.

5. Let there be R red balls and G green balls in the bag. Then the volunteers remove $\frac{2}{5}R$ red balls and $\frac{3}{7}G$ green balls. Since the same number of balls of each colour are removed, $\frac{2}{5}R = \frac{3}{7}G$. Rearranging gives $G = \frac{14}{15}R$. The fraction of balls contained in the bucket is $\dfrac{\frac{3}{5}R + \frac{4}{7}G}{R + G}$ and substituting for G gives $\dfrac{\frac{3}{5}R + \frac{8}{15}R}{R + \frac{14}{15}R}$. Cancelling R and simplifying shows that $\frac{17}{29}$ is the fraction of balls contained in the bucket.

Alternatively

Let n be the number of volunteers, so that n red balls and n green balls are removed. Hence the total number of red balls is $\frac{5}{2}n$ and the total number of green balls is $\frac{7}{3}n$, a total of $\frac{29}{6}n$. Therefore the number of balls contained in the bucket is $\frac{29}{6}n - 2n = \frac{17}{6}n$, that is, $\frac{17}{29}$ of all the balls.

Alternatively

If $\frac{2}{5}$ of the red balls are removed then this is equivalent to removing $\frac{6}{15}$ of the red balls. Similarly, if $\frac{3}{7}$ of the green balls are removed then this is equivalent to removing $\frac{6}{14}$ of the green balls. Now consider the diagram, showing the relative number of balls removed and remaining, in the correct proportion for each colour:

	Removed	Remaining	Total
Red	6	9	15
Green	6	8	14

Since the same numbers of red and green balls are removed, the units of proportion are equal. Hence the fraction of balls remaining is $\frac{9 + 8}{15 + 14} = \frac{17}{29}$.

6. Let n be the number of times the mathematician has filled the bottle from the tap. Consider the final mixture in the container. There is 1 litre of orange squash plus $\frac{1}{2}n$ litres of water, so the final mixture contains $1 + \frac{1}{2}n$ litres. Since 10% of the final mixture is 1 litre of orange squash, $\frac{1}{10}\left(1 + \frac{1}{2}n\right) = 1$. Hence $n = 18$, so the mathematician has filled the bottle 18 times.