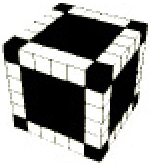
Solutions to the Olympiad Cayley Paper

1. The interior small cubes of each face of the original cube, shown grey in the diagram, are those with exactly one blue face; the interior small cubes along each edge, shown white in the diagram, are those with exactly two blue faces. If the original cube has sides of length $(n + 2)$ cm, then there are n^2 small cubes in each face with exactly one blue face, and n small cubes along each edge with exactly two blue faces.



Since the original cube has 6 faces and 12 edges, from the information given about the numbers of each type of cube, we have $6n^2 = 10 \times 12n$, that is $n^2 - 20n = 0$, and so $n(n - 20) = 0$. Hence $n = 0$ or $n = 20$. The case $n = 0$ is not possible since the original cube has edges greater than 2 cm in length. Therefore $n = 20$ and the edge length of the original cube is 22 cm.

2. Let the sides of the isosceles triangle have lengths s cm, t cm and t cm. Then the perimeter lengths of the triangle, parallelogram and rhombus are $(s + 2t)$ cm, $(2s + 2t)$ cm and $4t$ cm, respectively.

From the information given about the perimeters we get the equations $2s + 2t = s + 2t + 3$ and $4t = s + 2t + 7$. The first equation shows that $s = 3$; and then the second shows that $t = 5$. Hence the perimeter of the triangle is 13 cm.

3. Let $CX = 4x$ cm. Then $BX = 5x$ cm and $AB = 12x$ cm, since $BX : CX = 5 : 4$ and $AB = 3 \times CX$.

Now the area of triangle $CXA = \frac{1}{2} \times CX \times AB = 54$ cm^2 and hence $\frac{1}{2} \times 4x \times 12x = 54$. Rearranging gives $x^2 = \frac{9}{4}$, so that $x = \frac{3}{2}$ (since x is positive).

Noting that triangles CBA and XBA are right-angled, with sides in the ratio $3 : 4 : 5$ and $5 : 12 : 13$, respectively, we see that $AC = 15x$ cm and $AX = 13x$ cm. (These lengths may also be found using Pythagoras' theorem.)

Hence the length of the perimeter of triangle CXA is $4x + 13x + 15x$ cm, that is, 48 cm.

4. Since '$a679b$' is divisible by 36, it is also divisible by 4 and by 9. Now a number is divisible by 4 precisely when the number formed by the last two digits is a multiple of 4. Hence 4 divides '$9b$' and so $b = 2$ or $b = 6$. Similarly, a number is divisible by 9 precisely when the sum of the digits is a multiple of 9. Hence 9 divides $a + b + 22$.

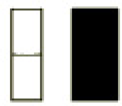
In the case when $b = 2$, we know 9 divides $a + 24$, so that $a = 3$. In the case when $b = 6$, we know 9 divides $a + 28$, so that $a = 8$. Therefore the possible numbers are 36792 and 86796.

5. Let $\angle OPQ = \alpha$. Since the length of PQ is equal to the radius of the circle and hence triangle OPQ is isosceles, it follows that $\angle QOP = \alpha$.

Applying the exterior angle theorem to triangle OPQ gives $\angle RQO = \angle QOP + \angle OPQ = 2\alpha$. Since OQ and OR are both radii of the circle and hence triangle OQR is isosceles, it follows that $\angle ORQ = 2\alpha$.

Now applying the exterior angle theorem to triangle OPR gives $\angle AOR = \angle OPR + \angle ORP = \alpha + 2\alpha = 3\alpha$. Therefore $\angle AOR = 3\angle BOQ$, as required.

6. Firstly, note that it is possible to tile both a 6×2 and a 6×3 rectangle, as shown on the right. It is clearly not possible to tile a 6×1 rectangle.



We claim that a $6 \times m$ rectangle may be tiled for any $m > 1$. If m is even, use repeated copies of the 6×2 tiling, as shown on the left below. If m is odd, use one copy of the 6×3 tiling and then use repeated copies of the 6×2 tiling, as shown on the right below.

Since a $6k \times m$ rectangle may be divided into k copies of a $6 \times m$ rectangle, it follows that it is possible to tile any $6k \times m$ rectangle, provided $m > 1$ and $k > 0$.