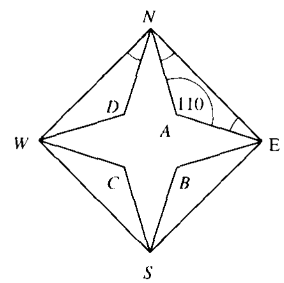
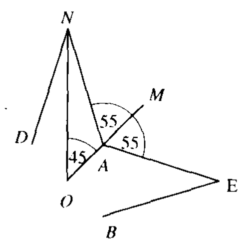
Solutions to the Olympiad Cayley Paper

1. We give a selection of the many possible solutions.

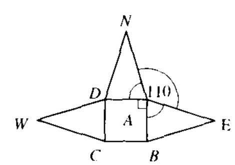
(a) From the symmetry, $NWSE$ is a square and angles ANE, AEN and DNW are equal. Considering triangle ANE, these angles are each 35°. Since $\angle WNE = 90°$, $\angle DNA = 20°$.

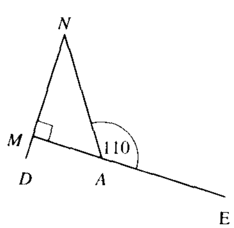




(b) Let O be the centre of the star and join O to N and to A, extending this line to M. Then OM and ON are lines of symmetry, so OM bisects $\angle NAE$ and $\angle NOA = 45°$. Now $\angle NAM$ is exterior to triangle NOA, so $\angle ONA = 10°$ and, from the symmetry, $\angle DNA = 20°$.

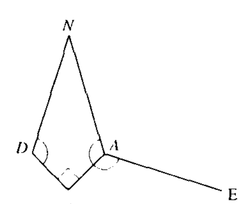
(c) From the symmetry, $ABCD$ is a square and $\angle DAN$ and $\angle BAE$ are equal. By considering angles at A, $\angle DAN = 80°$ and hence, from angles in a triangle, $\angle DNA = 20°$.





(d) Extend EA to meet ND at M. Now $\angle NMA = 90°$ from the rotational symmetry (rotating 90° anticlockwise about the centre moves AE to DN). Then, since $\angle NAE$ is an exterior angle of triangle NMA, $\angle DNA = 20°$.

(e) Join D and A to O, the centre of the star. From the symmetry, $\angle DOA = 90°$ and angles NAO, EAO and NDO are equal. These angles are each 125°, by considering angles at A. Now the sum of the angles in quadrilateral $NDOA$ is 360°, so $\angle DNA = 20°$.



(f) Let $\angle DNA$ equal $x°$. Then, by symmetry, the star octagon has four interior angles equal to $x°$ and four equal to 250° (using angles at a point). Now the sum of the interior angles of an octagon is $12 \times 90°$, so that $4x + 4 \times 250 = 1080$, giving $x = 20$ and so $\angle DNA = 20°$.

2. Let Mars, Venus and Pluto have ages m, v and p respectively. Then we have the three equations

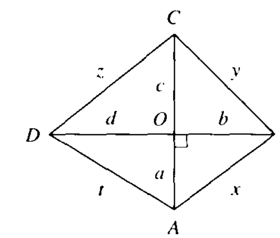
$$m + v + p = 192$$

$$m + p = 30 + v$$

$$v + p = 4 + m.$$

After subtraction, the first and second equations give $v = 162 - v$, so that $v = 81$. Similarly, the first and third equations give $m = 188 - m$, so that $m = 94$. Substituting these values into the first equation, we obtain $p = 17$. Hence Mars, Venus and Pluto are 94, 81 and 17 years old, respectively.

3. Let O be the point of intersection of the diagonals and let the lengths of OA, OB, OC and OD be a, b, c, and d respectively, as shown. Using Pythagoras' theorem in each of the right-angled triangles OAB, OBC, OCD and ODA, we obtain



$$x^2 = a^2 + b^2$$

$$y^2 = b^2 + c^2$$

$$z^2 = c^2 + d^2$$

$$t^2 = d^2 + a^2$$

from which $x^2 + z^2 = a^2 + b^2 + c^2 + d^2 = y^2 + t^2$, as required.

Note that the result remains true even if the quadrilateral ABCD is not convex.

4. (a) $1,\ 2,\ 2,\ 1,\ \frac{1}{2}$.

(b) The sequence is $1,\ 2,\ 2,\ 1,\ \frac{1}{2},\ \frac{1}{2},\ 1,\ 2,\ \ldots$ and so repeats every six terms. Hence the fiftieth term equals the second, which is 2.

(c) If the first two terms are a and b (neither of them zero), then the sequence is

$$a,\ b,\ \frac{b}{a},\ \frac{1}{a},\ \frac{1}{b},\ \frac{a}{b},\ a,\ b,\ \ldots$$

which also repeats every six terms. If $a = b = 1$ then the sequence is constant. *Note that if either of the first two terms is zero then the sequence is undefined.*

5. Team A lost all their matches, scoring 2 goals and having 3 goals scored against them. Since each lost match leads to at least one more 'goal against' a team than the number of 'goals for', team A can have played only one match, with a score of 2 - 3. Now B only had one goal scored against them and G did not win any matches, so A's opponents were C and the result was A 2 C 3.

Team G drew one match. Now neither A nor B drew any matches, so G's opponents were C, which completes C's list of two matches played. Team C scored 4 goals in all, of which 3 were against A, and hence the result of this match was C 1 G 1.

Team B played at least one match, since they had one goal scored against them, but can only have played against G and hence played exactly one match. Team G has 4 'goals against' unaccounted for, so the result was B 4 G 1.

The completed table is:

Team	Played	Won	Lost	Drawn	Goals for	Goals against
A	1	0	1	0	2	3
B	1	1	0	0	4	1
C	2	1	0	1	4	3
G	2	0	1	1	2	5

6. For ease of reference, label the columns 0 to 5 as shown. From column 5, we see that Y is even. Hence, since the greatest possible 'carry' is 1, looking at column 2 there can be no carry from column 3 and so $T < 5$. Also, comparing columns 2 and 5, A and S differ by 5. Hence neither A nor S equals 5.

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ & M & A & T & H & S \\ + & M & A & T & H & S \\ \hline C & A & Y & L & E & Y \end{array}$$

From column 0 we know that $M \geqslant 5$ and $C = 1$. Now consider columns 1 and 2. If $A > 5$, then there is a carry from column 2 to column 1, so that A is odd. If $A < 5$ then there is no such carry and A is even. Thus there are four cases to consider, with $A = 2, 4, 7$ or 9.

If $A = 9$, using the information found above we can start to complete the sum as shown. But now there is no possible value for M, since 9 is already allocated.

$$\begin{array}{cccccc} & ? & 9 & - & - & 4 \\ + & ? & 9 & - & - & 4 \\ \hline 1 & 9 & 8 & - & - & 8 \end{array}$$

If $A = 2$, we can proceed as far as shown, noting that $T = 3$ since this is the only unused digit below 5. But now there is no possible value for L, since 6 and 7 are already allocated.

$$\begin{array}{cccccc} & 6 & 2 & 3 & - & 7 \\ + & 6 & 2 & 3 & - & 7 \\ \hline 1 & 2 & 4 & ? & - & 4 \end{array}$$

If $A = 7$, we can proceed further, noting that $L = 6$ since 7 has already been used. But now there is no possible value for H, since all options lead to a carry to column 3.

$$\begin{array}{cccccc} & 8 & 7 & 3 & ? & 2 \\ + & 8 & 7 & 3 & ? & 2 \\ \hline 1 & 7 & 4 & 6 & - & 4 \end{array}$$

If $A = 4$, we can proceed as far as shown. There is a carry from column 5 so we conclude that E is odd and there are only two options, $E = 3$ or 5, since all other odd digits are allocated. Both cases lead to a solution:

$$\begin{array}{cccccc} & 7 & 4 & T & H & 9 \\ + & 7 & 4 & T & H & 9 \\ \hline 1 & 4 & 8 & L & E & 8 \end{array}$$

$$\begin{array}{cccccc} & 7 & 4 & 2 & 6 & 9 \\ + & 7 & 4 & 2 & 6 & 9 \\ \hline 1 & 4 & 8 & 5 & 3 & 8 \end{array} \qquad \begin{array}{cccccc} & 7 & 4 & 3 & 2 & 9 \\ + & 7 & 4 & 3 & 2 & 9 \\ \hline 1 & 4 & 8 & 6 & 5 & 8 \end{array}$$