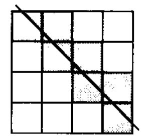
Solutions to the Olympiad Cayley Paper

Section A

A1 Numbering the trees 1 to 17, Basil marks 1, 3, 5, 7, 9, 11, 13, 15 and 17 on his way out, and 17, 14, 11, 8, 5 and 2 on his way back. So the five trees 4, 6, 10, 12 and 16 have no mark.

A2 The diagram to the right shows that you can cut seven squares. You cannot cut more because once the straight line has entered the grid, and so cut into one small square, it can enter other squares only by crossing an inside dividing line, either horizontally or vertically. There are only six of these, so it can cut only seven squares in all.



A3 The total value of the 5 parrots was 5 × €6000 = €30 000. After one has flown away, the total value is 4 × €5000 = €20 000. So the value of the parrot that escaped was €10000.

A4 If the original number is 19, then crossing out 9 leaves 1, and 1 × 19 = 19. If '*ab*' (meaning $10a + b$) is the original two-digit number, then crossing out b leaves a. If the original number was more than 19 times larger than a, then $10a + b > 19a$ leading to $b > 9a$. This is impossible because b is a single-digit number and a is not zero. The greatest possible value of x is 19.

A5 The factors of 42 are 2, 3 and 7. If the perimeter was 18 then its base was 2 by 7. Thus the height was 3 cm.

Section B

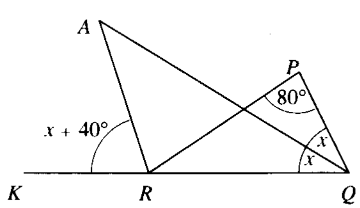
B1 The number '*sss*' equals $s \times 111 = s \times 3 \times 37$. Now 37 is prime, so one of the two numbers '*pq*' and '*rq*' is 37 or 74.

The case 74 is not possible, since then $q = 4$, giving $s = 6$ and so '*sss*' equals 9×74.

The case 37 gives $q = 7$, $s = 9$ and '*sss*' = 27 × 37, so that $p, r = 2, 3$, in either order.

B2 Let $\angle AQP = x$, so that $\angle AQR = x$ since AQ bisects $\angle PQR$.

Then $\angle KRP = 2x + 80°$, since $\angle KRP$ is an exterior angle of triangle PQR and so equals the sum of the opposite interior angles. But AR bisects $\angle KRP$, so that $\angle ARK = x + 40°$.



However, $\angle ARK = x + \angle RAQ$, since $\angle ARK$ is an exterior angle of triangle RAQ. It follows that $\angle RAQ = 40°$.

B3 It is possible to complete a table showing the numbers of pupils in each category, as shown.

	hockey	swimming	total
tennis	126	54	180
badminton	99	21	120
total	225		300

There are 300 children, 60% of them play tennis and 40% play badminton, so 180 play tennis and 120 play badminton. Also, 30% of the tennis players swim, so there are 54 tennis players who swim. Hence there are 180 − 54 = 126 tennis players who play hockey.

Now 56% of the hockey players also play tennis, so 56% of the number of hockey players = 126, therefore the number of hockey players = $\frac{126}{56} \times 100 = 225$.
Hence the number of hockey players who play badminton is 225 − 126 = 99 and so the number of badminton players who swim is 120 − 99 = 21.

B4 Let the circles with diameters PQ and RS have radius x and the circle with diameter QR have radius y. Then the radius of the circle with diameter PS is $y + 2x$, so that the shaded area is

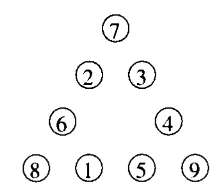
$$\begin{aligned}\tfrac{1}{2}\pi(y + 2x)^2 - \pi x^2 + \tfrac{1}{2}\pi y^2 &= \tfrac{1}{2}\pi(y^2 + 4xy + 4x^2) - \pi x^2 + \tfrac{1}{2}\pi y^2 \\ &= \tfrac{1}{2}\pi y^2 + 2\pi xy + 2\pi x^2 - \pi x^2 + \tfrac{1}{2}\pi y^2 \\ &= \pi y^2 + 2\pi xy + \pi x^2 \\ &= \pi(x + y)^2\end{aligned}$$

which is the area of a circle with radius $x + y$. But $MN = (y + 2x) + y = 2y + 2x$, so that the circle with diameter MN has radius $x + y$, as required.

[The shaded figure is known as a *salinon* and this result about its area appears in Archimedes, *Liber Assumptorum*, Proposition 14.]

B5 (a) The total of the numbers 1 to 9 is 45. Let the numbers placed at the vertices of the triangle be a, b and c. The numbers along each side of the triangle add up to T, so that adding the three sides together gives $3T$. This total includes all nine numbers, but with a, b and c included twice. So, $3T = 45 + a + b + c$. The smallest value for $a + b + c$ is $1 + 2 + 3 = 6$ and the largest is $7 + 8 + 9 = 24$. Hence $45 + 6 \leqslant 3T \leqslant 45 + 24$ and so $17 \leqslant T \leqslant 23$.

(b) We know from above that 7, 8 and 9 are placed at the vertices. The diagram shows one possible solution.



(c) Now, $a + b + c = 15$ and so the only possible choices for the values of a, b and c are: 9 5 1; 9 4 2; 8 6 1; 8 5 2; 8 4 3; 7 6 2; 7 5 3; 6 5 4. Some of these may lead to a solution and some may not, but we can conclude that there are at most these 8 possible choices.